Reliable Network Synthesis: Some Recent Developments

Wendy Myrvold [1]
University of Victoria

Abstract

Given n nodes and m links, what is the most reliable network topology? We survey some of the recent progress that has been made on this question for the all-terminal reliability model. As well, we suggest several interesting areas for future research.

1 Introduction

Colbourn surveys many of the combinatorial questions regarding network reliability in his monograph [6]. We focus on one of the more popular reliability models mentioned in this book, *all-terminal reliability* which we define shortly. The starting point for this research project was the survey of Boesch [2]. This paper surveys work up to 1986 for the problem of synthesizing reliable networks under the all-terminal reliability model. This work is a first attempt at updating the wealth of information found in this paper.

We model a network subject to component failure by an undirected probabilistic graph. Vertices represent communication centers, and edges represent connections between them. Edges operate independently at random with probability p. Because vertex failure implies failure of the network, we assume that vertices do not fail- not because this is realistic, but because it is computationally very easy to account for vertex failures if we know the reliability assuming that they do not fail. The network *fails* if it is not possible for all the vertices to communicate with each other (or in other words, if the failed edges

[1] Supported by NSERC.

form a cutset of the graph). Here we define a *cutset* to be any set of edges whose removal leaves the graph disconnected; some authors reserve this term for what we would call a minimal cutset. We use n for the number of vertices of a graph, and m for the number of edges.

The *reliability* of a graph G, $Rel(G)$, is the probability that G is connected. A standard formula for the reliability is:

$$Rel(G) \quad = \quad \sum_{i=0}^{m} N_i(G) \cdot p^i \cdot (1-p)^{m-i}, \tag{1}$$

where $N_i(G)$, or N_i when G is understood, denotes the number of connected spanning subgraphs of G with i edges. The *cut frequency vector* of a graph G has ith component equal to the number of cutsets of G of size i, equal to $\binom{m}{i} - N_{m-i}$.

We discuss the relevant issues regarding the synthesis of a most reliable network under this model. We first introduce the concept of uniformly-most reliable networks in Section 2. We show that the best graphs must have a maximum number of spanning trees, and a lexicographically minimum cut frequency vector, and these properties are addressed in Sections 3 and 4. The problem restricted to planar graphs is discussed in Section 5. We conclude in Section 6 with several interesting avenues of future research.

2 Uniformly-Most Reliable Networks

First, from this point onwards, we only compare the reliability of two graphs if they have the same order and size. The $(n, m)-$*graphs* are the collection of graphs having n vertices and m edges. Unless otherwise stated, we restrict our attention to simple graphs (those with no loops or multiple edges).

Kel'mans pointed out early on that the actual value of p could be a relevant factor when distinguishing between candidate topologies [11]. This point was reiterated by Boesch [2, pp. 346-347] who includes a simple example of two graphs whose reliability polynomials cross.

Instead of giving up on characterizing the best graphs, Boesch realized that in many situations, one topology stands out as being most reliable for all values of p. He called such a graph *uniformly optimally* *reliable*. We prefer the phrase *uniformly-most reliable* as it is not such a tongue twister.

Boesch conjectured in 1986 [2, p. 348]that uniformly-most reliable graphs always exist not realizing that Kel'man's [12] had much earlier (1981) published an infinite family of counterexamples. The same family was independently rediscovered [18] in 1991 in response to the conjecture of Boesch.

It has subsequently been shown that the all-terminal reliability polynomials of two graphs can cross twice [7]. Kel'mans has announced that he has a construction for multiple crossings, but his conference talk [13] only covered the 2-terminal construction in any detail.

The following two theorems are obvious observations about uniformly-most reliable graphs. Two corollaries are that uniformly-most reliable graphs must have a maximum number of spanning trees, and their cutset vectors must be lexicographically minimum.

Theorem 2.1 *If $N_i(G) = N_i(H)$ for $i = 0, 1, 2, ..., k$, and $N_{k+1}(G) > N_{k+1}(H)$ then G is more reliable than H for p close to zero (when the edges are very unreliable).*

Theorem 2.2 *If $N_i(G) = N_i(H)$ for $i = m, m-1, m-2, ..., m-k$, and $N_{m-k-1}(G) > N_{m-k-1}(H)$ then G is more reliable than H for p close to one (when the edges are very reliable).*

3 Maximizing Spanning Trees

As remarked earlier, maximizing the number of spanning trees is critical for ensuring that a network is most reliable when edges are sufficiently unreliable. Here we summarize the cases where the graphs with a maximum number of trees have been classified.

For graphs having up to $n+3$ edges, the problem has been solved. It has also been proved that these graphs are uniformly-most reliable by Boesch, Li and Suffel [3] for up to $n+2$ edges and by Wang [28] for $n+3$ edges. The best graphs are summarized in the following table:

m	Graphs with the most trees	Ref.
$n-1$	Any tree.	
n	An n-cycle.	
$n+1$	A θ-graph with path lengths as even as possible.	[29]
$n+2$	A particular subdivision of K_4.	[27]
$n+3$	A particular subdivision of $K_{3,3}$.	[3]

Wang and Wu [29] conjectured that starting with the 3-regular chordal graph on n vertices consisting of a Hamilton cycle together with chords connecting each vertex to the one directly opposite on the cycle, and subdividing edges appropriately would result in a family of graphs maximizing the number of trees. We know this works when the underlying 3-regular graph has up to 6 vertices ($K_{3,3}$ is isomorphic to this graph), and suspect that $n = 8$ is fine as well. However, the Petersen graph has more spanning trees than their proposed 10-vertex solution, so this pattern does not continue.

In a *regular complete multipartite graph*, the vertices can be partitioned into sets of equal size so that within a set, no two vertices are adjacent, and every two vertices chosen from two different sets are adjacent. A very interesting result obtained by eigenvalue techniques is that regular complete multipartite graphs maximize the number of spanning trees [5]. It is not known if these graphs are uniformly-most reliable.

The remaining cases where the problem has been solved are dense graphs (when at most $n-2$ edges are removed from the complete graph). Because the graphs we consider are close to being complete, it is often easier to consider the complements.

When the number of edges is at most $n/2$ less than that in the complete graph K_n, the best topology is the graph whose complement consists of a set of independent edges. This was proved independently by Kel'mans [15] and Shier [25]. It has also been shown that these graphs are uniformly-most reliable by using a reliability increasing operation called *swing surgery* [24] initially developed by Kel'mans [12]. Kel'mans announced [14] that he has characterized the remaining dense cases having up to n edges removed from K_n. Petingi, Boesch, and Suffel [21, 22] have proofs for all but a few small cases.

In the remaining cases, Gilbert and Myrvold [8] describe the best graphs assuming that they are almost-regular (as is every known optimal case). The simple algebraic proof technique they develop permits easy comparisons of the almost-regular graphs whose components are a union of paths and cycles.

The complements of the best almost-regular graphs are (P_k denotes a path on k vertices and C_k is a cycle on k vertices) [8]:

1. For $n/2 \leq m \leq n-2$, C_3's, and P_2's and either P_2 or P_3 or two P_3's.

2. For $m = n-1$, C_3's, and P_2's and either P_2 or P_3 or P_4.

3. For $m = n$, at most one of C_4 or C_5 plus C_3's.

Petingi, Boesch and Suffel [21, 22] show that these graphs have the maximum number of spanning trees as long as at most $n-2$ edges have been removed. They also handle the cases where $n = 3k$ edges are removed (k is an integer), or when $n-1$ edges are removed and $n = 3k+2$.

4 Cutsets

In most practical applications, the edges of a network are extremely reliable. As remarked earlier, the more reliable networks in this situation have a lexicographically minimum cut frequency vector. Two basic requirements are that λ, the size of a minimum cutset of G, should be maximized, and then N_λ should be minimized [11].

A graph G is *super-λ* if the only λ-edge cutsets are the incidence sets of vertices of minimum degree in G. For an r-regular graph, this implies that λ equals r, and N_λ equals n. A graph must be super-λ (if possible) to lexicographically minimize the cut frequency vector.

In both [4] and [1], circulant graphs are proposed as a reliable network topology partly because they are super-λ graphs. Recent work however has demonstrated that at least in the 3-regular case, these are perhaps one of the worst super-λ graphs to use [19, 20]. Certain generalized Petersen graphs are suggested as an alternative topology. This work pinpointed a close connection between the girth of a graph (the size of a smallest cycle) and the numbers of small cutsets, giving credence to the conjecture that a uniformly-most reliable network must maximize girth.

Other authors have considered the cut frequency vectors of cycle permutation graphs [23], and chordal rings [10]. Further study is definitely needed.

5 Planar Graphs

In previous work, no restrictions were placed on the networks constructed excepting the numbers of vertices and edges. In an actual application, it may not be feasible to construct certain topologies with the given technology. Formulating the requirements and using them to study a restricted version of the problem is more likely to yield results that are immediately applicable.

A graph is *planar* if it can be drawn on a flat surface without any edges crossing. Planar graphs are attractive to circuit designers etching metal circuits onto boards because crossing the lines would cause the circuit to short out. As a first step towards becoming more applied, we consider planar networks.

A natural starting point is to look for uniformly-most reliable planar graphs; the class of graphs under consideration is restricted to the (n, m)-graphs that are planar. Torrey Hoffman [9] using Brendan McKay's **makeg**, a graph generator based on the graph isomorphism program **nauty** [16] and software for reliability written by W. Myrvold (also used for [17]), computed the reliability polynomials of small simple planar graphs. An interesting multigraph (Figure 1(a)) was subsequently discovered by Sandra Trace [26]. It has fewer spanning trees than the best simple planar graph on the same number of edges (pictured in Figure 1(b)). But because it has no 3-edge cutsets (the simple graph has one), it is the preferred topology when edges are very reliable. This is the first example known where a multigraph is preferred over a simple graph.

6 Future Work

Several conjectures about uniformly-most reliable graphs are folklore and to date, no counterexamples have been found. These include assertions that a uniformly-most reliable network has degrees almost-regular, maximum girth, minimum diameter, and no multiple edges (if

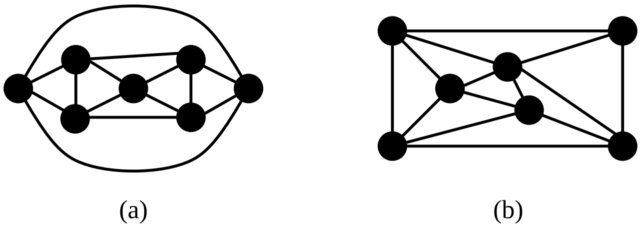

Figure 1: Two interesting planar graphs.

possible). Also, it has been conjectured that regular complete multipartite graphs are uniformly-most reliable. One interesting direction for future research is to try to prove some of these conjectures, possibly for restricted classes of graphs if the general problem is too hard.

As mentioned earlier, the restriction to planar graphs is a natural one motivated by applications. This is an area ripe for new results, as little has been done so far. For the applications where this restriction is not suitable, further communication is needed between the graph theorists and the builders of the networks to describe a theoretical model matching the practical restrictions.

Characterizing the graphs which are best with regards to the small cutsets is another area which has been under researched. The survey of Boesch [2, last sentence, p. 343] unfortunately gives the impression that further work on this question is unnecessary. In fact, we purport that it is even more relevant than maximizing spanning trees, as most practical applications have very reliable edges and we have seen that this means the critical factor is the numbers of small cutsets.

Continuing work on characterizing graphs with a maximum number of spanning trees is likely to prove increasingly difficult. One possibility is to extend the techniques used for dense graphs to determine the subgraphs whose removal from a regular complete bipartite (or multipartite) graph results in the remaining graph having a maximum number of trees. Although they may not maximize the number of trees overall, they should be pretty good because the regular complete multipartite graphs have a maximum number of trees. One major benefit of restricting to this class of graphs is that the algebra should prove manageable.

References

[1] F. Boesch and J. Wang. Super line-connectivity properties of circulant graphs. *SIAM Journal of Algebraic and Discrete Methods*, 7:89–98, 1986.

[2] F. T. Boesch. On unreliability polynomials and graph connectivity in reliable network synthesis. *Journal of Graph Theory*, 10:339–352, 1986.

[3] F. T. Boesch, X. Li, and C. Suffel. On the existence of uniformly optimally reliable networks. *Networks*, 21:181–194, 1991.

[4] F. T. Boesch and J. Wang. Reliable circulant networks with minimum transmission delays. *IEEE Transactions on Circuits and Systems*, 32:1286–1291, 1985.

[5] C. S. Cheng. Maximizing the total number of spanning trees in a graph; two related problems in graph theory and optimum design theory. *Journal Combinatorial Theory, series B*, 31:240–248, 1981.

[6] C. J. Colbourn. *The Combinatorics of Network Reliability*. Oxford University Press, New York, 1987.

[7] C. J. Colbourn, D. D. Harms, and W. J. Myrvold. Brief communication: Reliability polynomials can cross twice. *The Franklin Institute*, 330(3):629–633, 1993.

[8] B. Gilbert and W. Myrvold. Maximizing spanning trees in almost-complete graphs. *Networks*, 30:23–30, 1997.

[9] T. Hoffman. Course project for CSC 582: Network Reliability, 1995.

[10] X. Hu and F. Hwang. Reliabilities of chordal rings. *Networks*, 22:487–501, 1992.

[11] A. K. Kel'mans. Connectivity of probabilistic networks. *Automation and Remote Control – Translated from: Avtomatika i Telemekhanika (Russian)*, (3):98–116, March 1967.

[12] A. K. Kel'mans. On graphs with randomly deleted edges. *Acta Math. Acad. Sci. Hung.*, 37:77–88, 1981.

[13] A. K. Kel'mans. Network reliability functions can cross many times. Talk 163, 25th SE Conference on Combinatorics, Graph Theory and Computing, Boca Raton, Florida, March 7-11, 1994.

[14] A. K. Kel'mans. Laplacian polynomials and the number of spanning trees of a graph. Talk 207, 26th SE Conference on Combinatorics, Graph Theory and Computing, Boca Raton, Florida, March 6-10, 1995.

[15] A. K. Kel'mans and V. M. Chelnokov. A certain polynomial of a graph and graphs with an extremal number of trees. *Journal of Combinatorial Theory, series B*, 16:197–214, 1974.

[16] B. D. McKay. Practical graph isomorphism. *Proc. Tenth Manitoba Conf. Numerical Math. and Computing, Cong. Num.*, 30:45–87, 1981.

[17] W. Myrvold. Uniformly-most reliable graphs do not always exist. *Technical Report DCS-120-IR, Department of Computer Science, University of Victoria, Victoria, B.C., Canada*, 1989.

[18] W. Myrvold, K. Cheung, L. Page, and E. Perry. Uniformly-most reliable networks do not always exist. *Networks*, 21:417–419, 1991.

[19] J. Nadon. Beyond super-λ: counting 4-edge cutsets in 3-regular graphs. Master's thesis, Department of Computer Science, University of Victoria, Victoria, B.C., 1994.

[20] J. Nadon, W. Myrvold, and C. von Schellwitz. Circulants are the worst. In preparation, 1996.

[21] L. Petingi. *On the characterization of graphs with maximum number of spanning trees*. PhD thesis, Stevens Institute of Technology, Hoboken, N.J., 1991.

[22] L. Petingi, F. Boesch, and C. Suffel. On the characterization of graphs with maximum number of spanning trees. *Networks*, 179(1-3):185–203, 1998.

[23] B. Piazza and S. Stueckle. On the cut frequency vector of permutation graphs. *Congressus Numerantium*, 49:143–145, 1985.

[24] A. Satyanarayana, L. Schoppmann, and C. L. Suffel. A reliability-improving graph transformation with applications to network reliability. *Networks*, 22:209–216, 1992.

[25] D. R. Shier. Maximizing the number of spanning trees in a graph with n nodes and m edges. *Journal Research National Bureau of Standards, Section B*, 78:193–196, 1974.

[26] S. Trace. Course project for CSC 482: Network Reliability, 1995.

[27] S. S. Tseng and L. R. Wang. Maximizing the number of spanning trees of networks based on cycle basis representation. *International Journal of Computer Mathematics*, 28:47–56, 1989.

[28] G. Wang. A proof of Boesch's conjecture. *Networks*, 24:277–284, 1994.

[29] J. F. Wang and M. H. Wu. Network reliability analysis: on maximizing the number of spanning trees. *Proceeding of the National Science Council, Republic of China, Part A, Physical Science and Engineering*, 11(3):193–196, 1987.