Reliability of Planar Multigraphs

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Abstract

We consider all-terminal reliability, one of the more popular models of network reliability. We show that if you consider planar graphs on seven vertices and fourteen edges, the most reliable graph when edges are very reliable (as occurs in most practical applications) is a multigraph. This is the first example known where a multigraph is preferred over all simple graphs with the same order and size.

1 Network Reliability

We model a network by an undirected probabilistic graph. For the *all-terminal* model, edges operate independently at random with probability p and the network is *operational* if the underlying probabilistic graph is connected. Colbourn's monograph is an excellent starting point for a researcher interested in the combinatorics of network reliability [2].

The *reliability* of a graph G, $Rel(G)$, is the probability that G is connected. A standard formula for the reliability is:

$$Rel(G) = \sum_{i=0}^{m} N_i(G) \cdot p^i \cdot (1-p)^{m-i}, \tag{1}$$

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where $N_i(G)$ denotes the number of i-edge connected spanning subgraphs of G, and m is the number of edges. The *cut frequency vector* of a graph G has ith component, $i = 0, 1, ..., m$, equal to the number of cutsets of G of size i, equal to $\binom{m}{i} - N_{m-i}$.

For the synthesis problem, we are given n vertices, m edges, and the edge operation probability p, and are asked to produce a most reliable network. Boesch's survey [1] provides an introduction to this question for the all-terminal problem, and some recent updates can be found in [5].

In previous work, no restrictions were placed on the networks constructed excepting the numbers of vertices and edges. In an actual application, it may not be feasible to construct certain topologies with the given technology. Formulating the requirements and using them to study a restricted version of the problem is more likely to yield results that are immediately applicable.

A graph is *planar* if it can be drawn on a flat surface without any edges crossing. Planar graphs are attractive to circuit designers etching metal circuits onto boards because crossing the lines would cause the circuit to short out. As a first step towards becoming more applied, we consider planar networks.

A *simple* graph has no loops or multiple edges. For the general network synthesis problem, there are no examples known of a multigraph which is more reliable than every simple graph with the same order and size (assuming that it is possible to construct a simple graph). The main result of this paper is to demonstrate that for seven vertices and fourteen edges, the most reliable planar graph is a multigraph when edges are very reliable (as is the case in most practical applications).

2 The Main Theorem

We initially discovered that no 4-regular simple planar graphs exist by using **makeg**, a graph generator based on the graph isomorphism program **nauty** [4], to generate the graphs on seven vertices and fourteen edges. We tested planarity using an implementation of the algorithm of Demoucron et. al. [3] which was created for use as a component of a toroidality tester [6]. Here, we provide a theoretical proof that is much easier to verify.

Theorem 2.1 *There are no simple 7-vertex planar 4-regular graphs.*

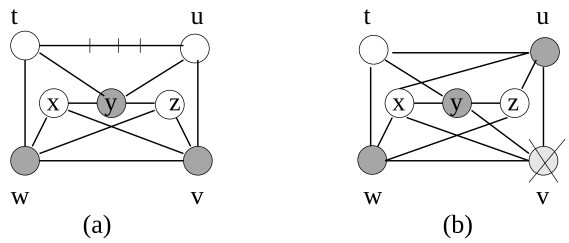

Figure 1: Pictures for the proof of Theorem 2.1

Proof. We assume that such a graph exists, and reach a contradiction by demonstrating that it must contain $K_{3,3}$, an obstruction to planarity. First, simple arithmetic shows that a planar graph on seven vertices and fourteen edges with no multiple edges must have eight faces with three edges and one face with four edges. Label the cycle which is the 4-face as t, u, v, w, and the remaining vertices as x, y, and z. If this 4-cycle has a chord, say (t, v), it is easy to see that this cannot be extended as there is no way to augment the graph retaining planarity so that vertices u and w have degree four without adding multiple edges. Thus, the subgraph induced by x, y and z contains exactly two edges (because the remaining twelve are incident to the other four vertices), which are without loss of generality (x, y) and (y, z).

There are two ways (up to isomorphism) to add two more edges incident to vertex y and we consider each in turn:

Case 1: The graph contains edges (t, y) and (u, y). Then because the graph is 4-regular with no multiple edges or chords of the 4-cycle, (v, x), (v, z), (w, x), and (w, z) are edges. But this induces a $K_{3,3}$ (consider Figure 1 (a) with edge (t, u) contracted) contradicting our assumption that the graph is planar.

Case 2: The graph contains edges (t, y) and (v, y). In this case, (u, x), (u, z), (w, x), and (w, z) are edges. This again induces a $K_{3,3}$ (see Figure 1 (b), ignoring vertex v). □

We are now ready to prove our main theorem.

Theorem 2.2 *The multigraph pictured in Figure 2 (a) is more reliable than any simple planar graph on the same order and size when edges are very reliable.*

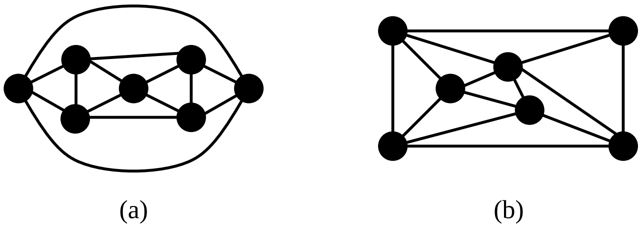

Figure 2: Two interesting planar graphs.

Proof. Because there are no simple planar 4-regular graphs (Theorem 2.1), any simple planar graph on seven vertices and fourteen edges must have a vertex of degree three or less. This means that there is at least one cut of size three or less. Thus because the graph pictured in Figure 2 (a) has minimum cut size equal to four, its cut frequency vector is lexicographically smaller than that of any simple alternative, and this is sufficient to show that it is more reliable when edges are very reliable. □

The graph pictured in Figure 2 (b) has more spanning trees than the multigraph in Figure 2 (a). Thus, there is no planar graph which is *uniformly-most reliable*, that is, more reliable for all values of p, as it is well known that such a graph must have a maximum number of spanning trees.

3 Future Research

The restriction of planarity matches some older technologies, but does not accurately reflect most current technology. In spite of this, studying planar topologies is an area which should be pursued further. This restricted model may in fact be more realistic than the widely studied problem of synthesizing a network with no constraints (other than the order and the size of the graph). As well, the tools and techniques developed will potentially prove applicable (perhaps with some modification) to more modern or realistic restrictions to the problem.

More communication between the network builders and the graph theorists is required to formulate sensible restrictions on the networks which are practical. Then for reasonable sets of restrictions, the characterization of uniformly-most reliable networks would provide valuable advice to network designers.

References

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