Design a PDA that accepts:

$L_1 = \{ a^p b^q c^r \;:\; p \neq q \}$

Hint:

L_1 is the union of two languages:

$L_2 = \{ a^p b^q c^r \;:\; p < q \}$

$L_3 = \{ a^p b^q c^r \;:\; p > q \}$

Closure Properties of Context-free Languages

Intersecting a context-free language and a regular language gives a context-free language.

Context-free languages are closed under union, concatenation and Kleene star.

Context-free languages are not closed under intersection or complement.

Thought question: are they closed under difference/exclusive or?

Theorem:

If L_1 is context-free and L_2 is regular then $L_1 \cap L_2$ is context-free.

Proof:

By construction. This proof is similar to the one on the assignment proving closure of regular languages under intersection.

$L = \{ w\ c\ w^R : w \in \{a, b\}^* \} \cap a^*\ c\ a^*$

Start state: s, Final State: {t}

State	Input	Pop	Next state	Push
s	a	ε	s	A
s	b	ε	s	B
s	c	ε	t	ε
t	a	A	t	ε
t	b	B	t	ε

Theorem:

Context-free languages are closed under union, concatenation and Kleene star.

Proof: By construction.

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and

let $G_2 = (V_2, \Sigma, R_2, S_2)$.

We show how to construct a grammar $G = (V, \Sigma, R\ S)$ for $L(G_1) \cup L(G_2)$, $L(G_1) \cdot L(G_2)$, and $L(G_1)^*$.

$L_1 = \{ a^n b^{2n} : n \geq 0 \}$

$S_1 \rightarrow a\, S_1\, bb$

$S_1 \rightarrow \varepsilon$

$L_2 = \{ u\, u^R\, v : u, v \text{ in } \{a, b\}^+ \}$

$S_2 \rightarrow U_2\, V_2$

$U_2 \rightarrow a\, U_2\, a$

$U_2 \rightarrow b\, U_2\, b$

$U_2 \rightarrow aa$

$U_2 \rightarrow bb$

$V_2 \rightarrow a\, V_2$

$V_2 \rightarrow b\, V_2$

$V_2 \rightarrow a$

$V_2 \rightarrow b$

$L_1 = \{ a^n b^{2n} : n \geq 0 \}$

$S_1 \rightarrow a\, S_1\, bb$

$S_1 \rightarrow \varepsilon$

UNION:

Start symbol S

$S \rightarrow S_1 \qquad S \rightarrow S_2$

$L_2 = \{ u\, u^R\, v : u, v \text{ in } \{a, b\}^+ \}$

$S_2 \rightarrow U_2\, V_2$

$U_2 \rightarrow a\, U_2\, a$

$U_2 \rightarrow b\, U_2\, b$

$U_2 \rightarrow aa$

$U_2 \rightarrow bb$

$V_2 \rightarrow a\, V_2$

$V_2 \rightarrow b\, V_2$

$V_2 \rightarrow a$

$V_2 \rightarrow b$

$L_1 = \{ a^n b^{2n} : n \geq 0 \}$

$S_1 \rightarrow a\, S_1\, bb$

$S_1 \rightarrow \varepsilon$

CONCATENATION:

Start symbol S

$S \rightarrow S_1\, S_2$

$L_2 = \{ u\, u^R\, v : u, v \text{ in } \{a, b\}^+ \}$

$S_2 \rightarrow U_2\, V_2$

$U_2 \rightarrow a\, U_2\, a$

$U_2 \rightarrow b\, U_2\, b$

$U_2 \rightarrow aa$

$U_2 \rightarrow bb$

$V_2 \rightarrow a\, V_2$

$V_2 \rightarrow b\, V_2$

$V_2 \rightarrow a$

$V_2 \rightarrow b$

KLEENE STAR:

Start symbol S

$S \rightarrow S_1 \; S$

$S \rightarrow \varepsilon$

$L_1 = \{ a^n b^{2n} : n \geq 0 \}$

$S_1 \rightarrow a \; S_1 \; bb$

$S_1 \rightarrow \varepsilon$

KLEENE STAR: Start symbol S

$S \to S_2\ S$

$S \to \varepsilon$

$L_2 = \{\ u\ u^R\ v : u, v \text{ in } \{a, b\}^+\}$

$S_2 \to U_2\, V_2$

$U_2 \to a\, U_2\, a$

$U_2 \to b\ U_2\, b$

$U_2 \to aa$

$U_2 \to bb$

$V_2 \to a\, V_2$

$V_2 \to b\, V_2$

$V_2 \to a$

$V_2 \to b$

Theorem: $L = \{ a^n b^n c^n \ : n \geq 0\}$ is not context-free.

We will prove this soon using the pumping theorem.

$L_1 = \{ a^p b^q c^r \ : p = q \}$

Design a PDA for L_1.

$L_2 = \{ a^p b^q c^r \ : p = r \}$

Design a context-free grammar for L_2.

This proves L_1 and L_2 are context-free.

$L_1 \cap L_2 = \{ a^n b^n c^n \ : n \geq 0 \}$.

Therefore, context-free languages are not closed under intersection.

$L = \{ a^n b^n c^n : n \geq 0\}$

$L_1 = \{ a^p b^q c^r : p \neq q \}$

$L_2 = \{ a^p b^q c^r : p \neq r \}$

$L_3 = \{ a^p b^q c^r : q \neq r \}$

The complement of L is NOT equal to

$L_1 \cup L_2 \cup L_3$.

Which strings are in the complement of L but not in $L_1 \cup L_2 \cup L_3$?

$L = \{ a^n b^n c^n : n \geq 0 \}$

$L_1 = \{ a^p b^q c^r : p \neq q \}$

$L_2 = \{ a^p b^q c^r : p \neq r \}$

$L_3 = \{ a^p b^q c^r : q \neq r \}$

$L_4 = \{ w \in \{a, b, c\}^* : w \notin a^* b^* c^* \}$

$L_1 \cup L_2 \cup L_3 \cup L_4$ is context-free and is the complement of L.

Therefore, context-free languages are not closed under complement.

(a) If $x \notin L_1$ and $y \notin L_2$ then $x \cdot y \notin L_1 \cdot L_2$.

(b) A regular language can contain a subset which is not a regular language.

(c) The set ϕ^* does not contain any strings.

(d) If $L = \{ w \in \{ a, b \}^* : w = a^n b^n,\ n \geq 0 \}$, then $\overline{L} = \{ w \in \{ a, b \}^* : w = a^n b^m, n > m\}$

$\cup \{ w \in \{ a, b \}^* : w = a^n b^m, n < m\}$.

Critique this PDA for

$L = \{ u u^R v v^R : u \in \{0,1\}^* \text{ and } v \in \{0,1\}^+ \}$

s	ε	ε	t	ε
t	0	ε	t	0
t	1	ε	t	1
t	0	0	t	ε
t	1	1	t	ε
t	ε	ε	u	ε

u	0	ε	u	0
u	1	ε	u	1
u	ε	ε	v	ε
v	0	0	v	ε
v	1	1	v	ε
v	ε	ε	f	ε

After you identify the bugs and inefficiencies, design a nicer PDA for this language that is correct.