Let L=

$\{ w= u\,v : u \in \{a, b\}^*, v \in \{c, d\}^* \text{ and } |u|= |v|\}$

1. Design a context-free grammar that generates L.
2. Use your grammar and the construction from last class to design a PDA that accepts L.

No class or tutorial on Tuesday July 3: reading break is July 2-3 for May to August classes.

During reading break: try doing problems on old midterm and final exams. Our final exam is fast approaching.

The worst question on the midterm was Question #4. These were on previous midterms. Make sure you know how to work all the questions on the old final exams and ask about them at the final exam tutorial if you are unsure of the answers.

Results for Question #4:

Number Correct	Number of students
0	13
1	14
2	2
3	1
4	1

(a) If $x \notin L_1$ and $y \notin L_2$ then $x \cdot y \notin L_1 \cdot L_2$.

(b) A regular language can contain a subset which is not a regular language.

(c) The set $\phi*$ does not contain any strings.

(d) If $L = \{ w \in \{ a, b \}^* : w = a^n b^n, \ n \geq 0 \}$,
then $\overline{L} = \{ w \in \{ a, b \}^* : w = a^n b^m, n > m\}$
$\cup \{ w \in \{ a, b \}^* : w = a^n b^m, n < m\}$.

To organize your cases for the pumping lemma question, group according to first/last symbol of y:

Case	First	Last	x	y	z	Conditions
1	a	a	$(ab)^i$	$(ab)^j a$	$b(ab)^{p-i-j-1}\ c^{2p}$	$j \geq 0$
2	a	b	$(ab)^i$	$(ab)^j$	$(ab)^{p-i-j}\ c^{2p}$	$j \geq 1$
3	b	a	$(ab)^i a$	$b(ab)^j a$	$b(ab)^{p-i-j-2}\ c^{2p}$	$j \geq 0$
=	b	a	$(ab)^i a$	$(ba)^j$	$b(ab)^{p-i-j-1}\ c^{2p}$	$j \geq 1$
4	b	b	$(ab)^i a$	$b(ab)^j$	$(ab)^{p-i-j-1}\ c^{2p}$	$j \geq 0$

What happens if there is some c's in x or in y?

$$L = \{ (ab)^n c^m \ : \ n \leq m \leq 3n \}$$

$$w = (ab)^p c^{2p} \text{ where } 2p \geq k$$

Closure Properties of Context-free Languages

Intersecting a context-free language and a regular language gives a context-free language.

Context-free languages are closed under union, concatenation and Kleene star.

Context-free languages are not closed under intersection or complement.

Thought question: are they closed under difference/exclusive or?

Theorem:

If L_1 is context-free and L_2 is regular then $L_1 \cap L_2$ is context-free.

Proof:

By construction. This proof is similar to the one on the assignment proving closure of regular languages under intersection.

$L = \{ w\, c\, w^R : w \in \{a, b\}^* \} \cap a^*\ c\ a^*$

Start state: s, Final State: {t}

State	Input	Pop	Next state	Push
s	a	ε	s	A
s	b	ε	s	B
s	c	ε	t	ε
t	a	A	t	ε
t	b	B	t	ε

Theorem:

Context-free languages are closed under union, concatenation and Kleene star.

Proof: By construction.

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and

let $G_2 = (V_2, \Sigma, R_2, S_2)$.

We show how to construct a grammar $G= (V, \Sigma, R\ S)$ for $L(G_1) \cup L(G_2)$, $L(G_1) \cdot L(G_2)$, and $L(G_1)^*$.

$L_1 = \{ a^n b^{2n} : n \geq 0 \}$

$S_1 \rightarrow a S_1 bb$

$S_1 \rightarrow \varepsilon$

$L_2 = \{ u u^R v : u, v \text{ in } \{a, b\}^+ \}$

$S_2 \rightarrow U_2 V_2$

$U_2 \rightarrow a U_2 a$

$U_2 \rightarrow b U_2 b$

$U_2 \rightarrow aa$

$U_2 \rightarrow bb$

$V_2 \rightarrow a V_2$

$V_2 \rightarrow b V_2$

$V_2 \rightarrow a$

$V_2 \rightarrow b$

$L_1 = \{ a^n b^{2n} : n \geq 0 \}$

$S_1 \rightarrow a S_1 bb$

$S_1 \rightarrow \varepsilon$

UNION:

Start symbol S

$S \rightarrow S_1 \qquad S \rightarrow S_2$

$L_2 = \{ u u^R v : u, v \text{ in } \{a, b\}^+ \}$

$S_2 \rightarrow U_2 V_2$

$U_2 \rightarrow a U_2 a$

$U_2 \rightarrow b U_2 b$

$U_2 \rightarrow aa$

$U_2 \rightarrow bb$

$V_2 \rightarrow a V_2$

$V_2 \rightarrow b V_2$

$V_2 \rightarrow a$

$V_2 \rightarrow b$

$L_1 = \{ a^n b^{2n} : n \geq 0\}$

$S_1 \rightarrow a\ S_1\ bb$

$S_1 \rightarrow \varepsilon$

CONCATENATION:

Start symbol S

$S \rightarrow S_1\ S_2$

$L_2 = \{ u\ u^R\ v : u, v \text{ in } \{a, b\}^+\}$

$S_2 \rightarrow U_2\ V_2$

$U_2 \rightarrow a\ U_2\ a$

$U_2 \rightarrow b\ U_2\ b$

$U_2 \rightarrow aa$

$U_2 \rightarrow bb$

$V_2 \rightarrow a\ V_2$

$V_2 \rightarrow b\ V_2$

$V_2 \rightarrow a$

$V_2 \rightarrow b$

KLEENE STAR:

Start symbol S

$S \rightarrow S_1 \; S$

$S \rightarrow \varepsilon$

$L_1 = \{ a^n b^{2n} : n \geq 0 \}$

$S_1 \rightarrow a \; S_1 \; bb$

$S_1 \rightarrow \varepsilon$

KLEENE STAR: Start symbol S

$S \rightarrow S_2\ S$

$S \rightarrow \varepsilon$

$L_2 = \{\ u\ u^R\ v : u, v \text{ in } \{a, b\}^+\}$

$S_2 \rightarrow U_2\, V_2$

$U_2 \rightarrow a\, U_2\, a$

$U_2 \rightarrow b\ U_2\, b$

$U_2 \rightarrow aa$

$U_2 \rightarrow bb$

$V_2 \rightarrow a\, V_2$

$V_2 \rightarrow b\, V_2$

$V_2 \rightarrow a$

$V_2 \rightarrow b$

Theorem: $L = \{ a^n b^n c^n : n \geq 0\}$ is not context-free.

Proof: Next class using the pumping theorem for context-free languages.

Today: assume it is true.

$L_1 = \{ a^p b^q c^r : p = q \}$

Design a context-free grammar for L_1.

$L_2 = \{ a^p b^q c^r : p = r \}$

Design a PDA for L_2.

This proves L_1 and L_2 are context-free.

$L_1 \cap L_2 = \{ a^n b^n c^n : n \geq 0 \}$.

Therefore, context-free languages are not closed under intersection.

$L = \{ a^n b^n c^n \ : n \geq 0\}$

$L_1 = \{ a^p b^q c^r \ : p \neq q \}$

$L_2 = \{ a^p b^q c^r \ : p \neq r \}$

$L_3 = \{ a^p b^q c^r \ : q \neq r \}$

The complement of L is NOT equal to

$L_1 \cup L_2 \cup L_3$.

Which strings are in the complement of L but not in $L_1 \cup L_2 \cup L_3$?

$L = \{ a^n b^n c^n : n \geq 0 \}$

$L_1 = \{ a^p b^q c^r : p \neq q \}$

$L_2 = \{ a^p b^q c^r : p \neq r \}$

$L_3 = \{ a^p b^q c^r : q \neq r \}$

$L_4 = \{ w \in \{a, b, c\}^* : w \notin a^* b^* c^* \}$

$L_1 \cup L_2 \cup L_3 \cup L_4$ is context-free and is the complement of L.

Therefore, context-free languages are not closed under complement.