$$\boldsymbol{a} = (a_0, a_1, a_2, \ldots)$$

ordinary generating function of a:

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n \geq 0} a_n x^n$$

exponential generating function of a:

$$a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + \cdots = \sum_{n \geq 0} a_n \frac{x^n}{n!}$$

Many others, not as important.

What is the point?

"Natural" algebraic operations on generating functions have combinatorial significance, so we can transform combinatorics into algebra (and vice versa).

Notation:

$$\begin{aligned} \mathbb{N} &= \{0, 1, 2, \ldots\} \\ \mathbb{P} &= \{1, 2, 3, \ldots\} \\ [\boldsymbol{n}] &= \{1, 2, \ldots, n\} \\ [\boldsymbol{x}^{\boldsymbol{n}}] \sum a_k x^k &= a_n. \end{aligned}$$

Some operations:

$$\sum a_n x^n + \sum b_n x^n = \sum (a_n + b_n) x^n$$

$$\left(\sum a_n x^n\right)\left(\sum b_n x^n\right) = \sum c_n x^n,$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

$$\left(\sum a_n \frac{x^n}{n!}\right)\left(\sum b_n \frac{x^n}{n!}\right) = \sum c_n \frac{x^n}{n!},$$

where $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$.

Define

$$G(x) = \mathbf{1/F(x)}$$

if $F(x)G(x) = 1$ (exists if and only if $F(0) \neq 0$). E.g,

$$\frac{1}{1-ax} = 1 + ax + a^2x^2 + \cdots.$$

Let

$$F(x) = \sum_{n\geq 0} a_n x^n, \;\; G(x) = \sum_{n\geq 1} b_n x^n$$

(so $G(0) = 0$). Define the **composition** $F(G(x))$ by

$$F(G(x)) = \sum_{n\geq 0} a_n G(x)^n.$$

Makes sense **formally** since computing $[x^n]F(G(x))$ involves only a **finite** sum.

Examples. Let $G(0) = 0$. Then

$$\begin{aligned} e^{G(x)} &= \sum_{n\geq 0} \frac{G(x)^n}{n!} \\ -\log(1 - G(x)) &= \sum_{n\geq 1} \frac{G(x)^n}{n}. \end{aligned}$$

Lifting principle: All "familiar" formulas for convergent power series continue to hold whenever they make sense formally. E.g., if $G(0) = 0$ then

$$\begin{aligned} \log(e^{G(x)}) &= G(x) \\ e^{\log(1+G(x))} &= 1 + G(x). \end{aligned}$$

Sets. Let $n \in \mathbb{N}$ and

$$\boldsymbol{F_n}(\boldsymbol{x}) = \sum_{T \subseteq [n]} \prod_{i \in T} x_i,$$

a "list" of all subsets of $[n]$. E.g.,

$$F_2(\boldsymbol{x}) = 1 + x_1 + x_2 + x_1 x_2.$$

Since for each $i \in S$ either $i \in T$ or $i \notin T$, we have

$$F_n(\boldsymbol{x}) = (1+x_1)(1+x_2)\cdots(1+x_n).$$

Define

$$\boldsymbol{\binom{n}{k}} = \#\{T \subseteq S \,:\, \#T = k\}.$$

Put each $x_i = x$ to get

$$(1+x)^n = \sum_{k \geq 0} \binom{n}{k} x^k.$$

Illustrates technique of "late specialization."

Multisets. A **multiset** M on a set S is a set **with repeated elements** from S. E.g,

$$\{1,1,1,2,4,4,4,7,7\} = \{1^3,2,4^3,7^2\}$$

is a multiset on $[10]$. Let

$$\boldsymbol{\nu_M(i)} = \#\ i\text{'s in } M.$$

Let

$$\boldsymbol{G_n}(\boldsymbol{x}) = \sum_{M \text{ on } [n]} \prod_{i=1}^{n} x_i^{\nu_M(i)},$$

a "list" of all multisets on $[n]$. E.g.,

$$\begin{aligned} G_2(\boldsymbol{x}) &= 1 + x_1 + x_2 + x_1^2 + x_1x_2 + x_2^2 + \cdots \\ &= (1 + x_1 + x_1^2 + \cdots)(1 + x_2 + x_2^2 + \cdots) \\ &= \frac{1}{(1-x_1)(1-x_2)}. \end{aligned}$$

In general,

$$G_n(\boldsymbol{x}) = \frac{1}{(1-x_1)(1-x_2)\cdots(1-x_n)}.$$

Let $\left(\!\binom{\boldsymbol{n}}{\boldsymbol{k}}\!\right)$ denote the number of k-element multisets on $[n]$. E.g., $\left(\!\left(\binom{3}{2}\right)\!\right) = 6$:

$$11 \quad 22 \quad 33 \quad 12 \quad 13 \quad 23$$

Put $x_i = x$ to get

$$\sum_{k \geq 0} \left(\!\binom{n}{k}\!\right) x^k = \frac{1}{(1-x)^n}$$
$$= \sum_{k \geq 0} \binom{-n}{k} (-x)^k,$$

where

$$\binom{\boldsymbol{t}}{\boldsymbol{k}} = \frac{t(t-1)\cdots(t-k+1)}{k!}.$$

$$\sum_{k\geq 0} \left(\!\!\binom{n}{k}\!\!\right) x^k = \frac{1}{(1-x)^n}$$

$$= \sum_{k\geq 0} \binom{-n}{k} (-x)^k,$$

Hence

$$\left(\!\!\binom{n}{k}\!\!\right) = (-1)^k \binom{-n}{k} = \binom{n+k-1}{k}$$

(example of **reciprocity**).

Combinatorial or **bijective** proof that

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$$

Let

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n$$

be a k-multiset on $[n]$. Let $b_i = a_i + i - 1$. Then

$$1 \leq b_1 < b_2 < \cdots < b_k \leq n + k - 1,$$

and conversely (i.e., $a_i = b_i - i + 1$). Thus

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}.$$

RATIONAL GENERATING FUNCTIONS

A generating function $F(x) = \sum a_n x^n$ is **rational** if there are polynomials $P(x), Q(x)$ such that

$$F(x) = \frac{P(x)}{Q(x)},$$

i.e., $F(x)Q(x) = P(x)$. Can assume $Q(0) = 1$.

E.g.,

$$\sum_{n\geq 0} a^n x^n = \frac{1}{1-ax}.$$

More generally,

$$\frac{1}{(1-ax)^d} = \sum_{n\geq 0} \binom{-d}{n} (-ax)^n$$

$$= \sum_{n\geq 0} \binom{n+d-1}{d-1} a^n x^n.$$

Note: $\binom{n+d-1}{d-1}$ is a polynomial in n of degree $d-1$.

Fundamental theorem on rational generating functions.

Fix $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$, $\alpha_d \neq 0$.
Let $f : \mathbb{N} \to \mathbb{C}$. *TFAE:*

- $\sum_{n \geq 0} f(n)x^n = P(x)/Q(x)$,
  where $Q(x) = 1 + \alpha_1 x + \cdots + \alpha_d x^d$,
  $$P(x) \in \mathbb{C}[x],$$
  $$\deg(P) < \deg(Q) = d.$$
- *For all* $n \geq 0$,
  $$f(n+d)+\alpha_1 f(n+d-1)+\cdots+\alpha_d f(n) = 0$$
  (linear recurrence with constant coefficients).

- *For all* $n \geq 0$,

$$f(n) = \sum_{i=1}^{k} P_i(n)\gamma_i^n,$$

where

$$1+\alpha_1 x+\cdots+\alpha_d x^d = \prod_{i=1}^{k}(1-\gamma_i x)^{d_i},$$

the γ_i*'s are distinct, and*

$$P_i(n) \in \mathbb{C}[n], \quad \deg(P_i) < d_i.$$

Idea of proof. Use partial fractions to write $P(x)/Q(x)$ as linear combination of terms $(1-\gamma_i x)^e$, $e < d_i$.

What if $\deg P \geq \deg Q$? Then write (uniquely)

$$\frac{P(x)}{Q(x)} = L(x) + \frac{R(x)}{Q(x)},$$

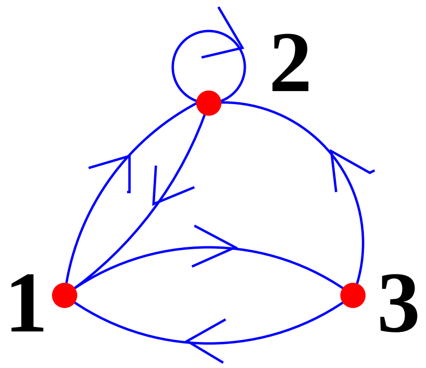
where $L(x), R(x) \in \mathbb{C}[x]$ and

$$\deg R(x) < \deg Q(x).$$

Thus $L(x)$ records the "exceptional values" (finitely many) where the fundamental theorem fails.

Example (the **transfer-matrix method**). Let $\boldsymbol{f(n)}$ be the number of sequences $a_1 \cdots a_n$, $a_i = 1, 2, 3$, with no $a_i a_{i+1} = 11$ or 23. Thus

$$f(n) = \#\text{ paths of length } n-1 \text{ in:}$$



Adjacency matrix: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$

Thus $\left(A^k\right)_{ij}$ is the number of paths of length k from i to j, so

$$f(n) = \sum_{i,j=1}^{3} \left(A^{n-1}\right)_{ij}.$$

$$\Rightarrow \sum_{n\geq 0} f(n+1)x^n = \sum_{i,j=1}^{3} \left(\sum_{n\geq 0} A^n x^n\right)_{ij}$$

$$= \sum_{i,j=1}^{3} (I - xA)^{-1}_{ij}.$$

Let $\boldsymbol{(B : j, i)}$ denote the matrix B with row j and column i removed. Then

$$B^{-1}_{ij} = (-1)^{i+j} \frac{\det(B; j, i)}{\det(B)},$$

so

$$\sum_{n \geq 0} f(n+1)x^n = \frac{\sum (-1)^{i+j} \det(I - xA : j, i)}{\det(I - xA)}$$

$$= \boldsymbol{\frac{3 + x - x^2}{1 - 2x - x^2 + x^3}}.$$

EXPONENTIAL GENERATING FUNCTIONS

Given $f : \mathbb{N} \to \mathbb{C}$, write

$$\boldsymbol{E_f(x)} = \sum_{n \geq 0} f(n) \frac{x^n}{n!}.$$

Proposition. *Given $f, g : \mathbb{N} \to \mathbb{C}$, define $h : \mathbb{N} \to \mathbb{C}$ by*

$$h(\#X) = \sum_{(S,T)} f(\#S) g(\#T),$$

where $\#X < \infty$ and $S, T \subseteq X$ such that

$$S \cup T = X, \quad S \cap T = \emptyset.$$

Then

$$E_f(x) E_g(x) = E_h(x).$$

Proof. Let $\#X = n$. There are $\binom{n}{k}$ pairs (S, T) with $\#S = k$ and $\#T = n - k$. Hence

$$h(n) = \sum_{k=0}^{n} \binom{n}{k} f(k) g(n-k)$$

$$= \left[\frac{x^n}{n!}\right] E_f(x) E_g(x). \quad \square$$

Example. Find the number $h(n)$ of ways to let $[n] = S \cup T$ with $S \cap T = \emptyset$, choose a subset of S, and choose an element of T. Here $f(n) = 2^n$ and $g(n) = n$. Thus

$$E_f(x) = \sum_{n \geq 0} 2^n \frac{x^n}{n!} = e^{2x}$$

$$E_g(x) = \sum_{n \geq 0} n \frac{x^n}{n!} = xe^x$$

$$\begin{aligned} \Rightarrow E_h(x) &= xe^{3x} \\ &= \sum_{n \geq 0} n3^{n-1} \frac{x^n}{n!}, \end{aligned}$$

whence $h(n) = \boldsymbol{n3^{n-1}}$.

Iterate previous proposition:

Proposition. *Fix* $k \in \mathbb{P}$ *and* $f_1, \ldots, f_k : \mathbb{N} \to \mathbb{C}$. *Define* $h : \mathbb{N} \to \mathbb{C}$ *by*

$$h(\#X) = \sum f_1(\#S_1) \cdots f_k(\#S_k),$$

where $\cup S_i = X$ *and* $S_i \cap S_j = \emptyset$ *if* $i \neq j$. *Then*

$$E_h(x) = E_{f_1}(x) \cdots E_{f_k}(x).$$

A **partition** of a finite set S is a collection $\{B_1, \ldots, B_k\}$ of subsets (called **blocks**) of S such that

$$\cup B_i = S, \quad B_i \neq \emptyset, \quad B_i \cap B_j = \emptyset \text{ if } i \neq j.$$

Write $\boldsymbol{\Pi(S)}$ for the set of partitions of S.

Partitions of $[3]$:

$$1-2-3 \quad 12-3 \quad 13-2 \quad 1-23 \quad 123$$

Exponential formula. *Given* $f : \mathbb{P} \to \mathbb{C}$, *define* $h : \mathbb{N} \to \mathbb{C}$ *by*

$$\begin{aligned} h(0) &= 1 \\ h(\#S) &= \sum_{\pi} f(\#B_1) \cdots f(\#B_k), \quad \#S > 0, \end{aligned}$$

where $\pi = \{B_1, \ldots, B_k\} \in \Pi(S)$. *Then*

$$\boldsymbol{E_h(x) = e^{E_f(x)}}.$$

Proof. Set $f(0) = 0$. For **fixed** k let

$$g_k(\#S) = \sum_{(B_1,\ldots,B_k)} f(\#B_1) \cdots f(\#B_k),$$

where $\{B_1, \ldots, B_k\} \in \Pi(S)$. Thus

$$E_{g_k}(x) = E_f(x)^k.$$

Since $T_i \neq \emptyset$, all $k!$ orderings of $T_1, \ldots, T_k$ are distinct. Thus for fixed k, if

$$h_k(\#S) = \sum_{\{B_1,\ldots,B_k\} \in \Pi(S)} f(\#B_1) \cdots f(\#B_k),$$

then $E_{h_k}(x) = \frac{1}{k!} E_{g_k}(x) = \frac{1}{k!} E_f(x)^k$. Hence

$$\begin{aligned} E_h(x) &= 1 + \sum_{k \geq 1} E_{h_k}(x) \\ &= \sum \frac{E_f(x)^k}{k!} = e^{E_f(x)}. \ \square \end{aligned}$$

Examples. (a) Let $\boldsymbol{\Pi_n} = \Pi([n])$ and $\boldsymbol{B(n)} = \#\Pi_n$ (**Bell number**). If $f(i) = 1$ $\forall i$ then

$$B(n) = \sum_{\{B_1,\ldots,B_k\}\in\Pi_n} f(\#B_1)\cdots f(\#B_k).$$

Thus

$$\begin{aligned}\sum_{n\geq 0} B(n)\frac{x^n}{n!} &= \exp \sum_{n\geq 1}\frac{x^n}{n!}\\ &= \exp(e^x - 1),\end{aligned}$$

(b) Let $f(n)$ be the number of **connected** graphs on the vertex set $[n]$. Thus $h(n)$ is the **total** number of graphs on $[n]$, so $h(n) = 2^{\binom{n}{2}}$. Hence

$$\sum_{n\geq 1} f(n)\frac{x^n}{n!} = \log \sum_{n\geq 0} 2^{\binom{n}{2}}\frac{x^n}{n!}.$$

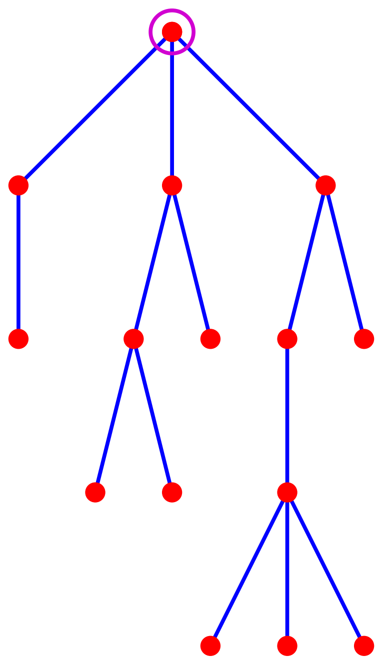
(Note that these series **diverge** for all $x \neq 0$.)

(c) Let $t_k(n)$ be the number of permutations w of $[n]$ satisfying $w^k = 1$. Thus every cycle length d of w satisfies $d|k$. We can choose w by partitioning $[n]$ into blocks of sizes $d|k$ and placing a cycle on each such block in $(d-1)!$ way. Hence

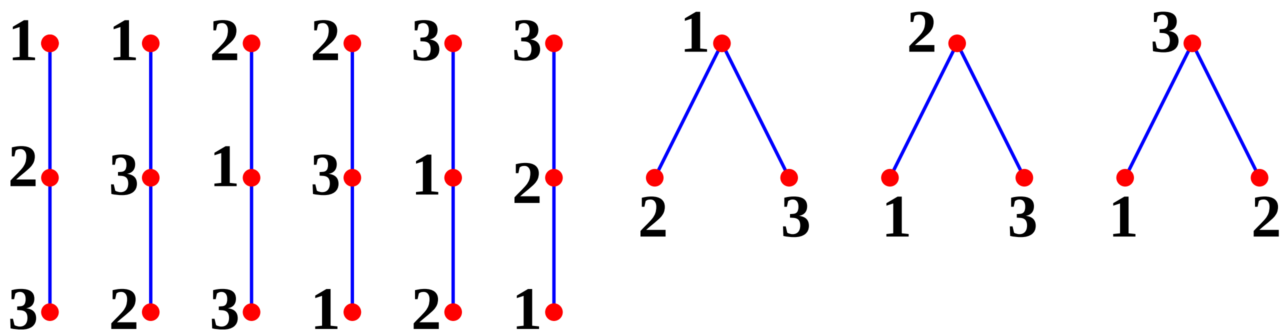
$$\begin{aligned}\sum_{n\geq 0} t_k(n)\frac{x^n}{n!} &= \exp \sum_{d|k}(d-1)!\frac{x^d}{d!} \\ &= \exp \sum_{d|k} \frac{x^d}{d}.\end{aligned}$$

TREES

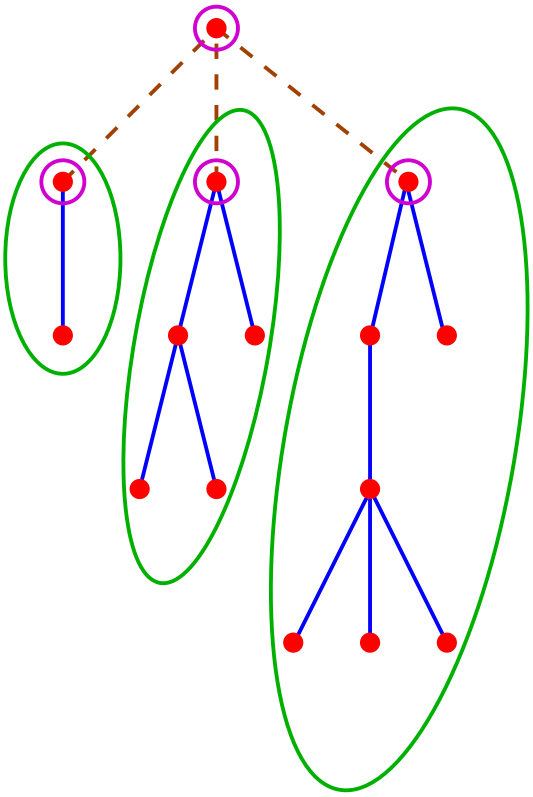
A **rooted tree** is a connected graph without cycles with one distinguished vertex (the **root**).



Let $r(n)$ be the number of rooted trees on the vertex set $[n]$. E.g., $r(3) = 9$:



To obtain a rooted tree T on $[n]$, choose a root r in n ways, choose a partition $\pi \in \Pi([n] - \{r\})$, place a rooted tree T_i on each block of π, and "join" r to the roots of each T_i.



Let

$$\boldsymbol{R(x)} = \sum_{n\geq 1} r(n)\frac{x^n}{n!}$$

$$e^{R(x)} = \sum_{n\geq 0} \boldsymbol{f(n)}\frac{x^n}{n!}.$$

Thus $f(n)$ is the number of **forests of rooted trees** on $[n]$, so $xe^{R(x)}$ is the exponential generating function for choosing a 1-element subset of $[n]$ (the root) and placing a forest of rooted trees on the remaining elements. Since this structure is equivalent to a rooted tree on $[n]$, we have

$$\boldsymbol{R(x) = xe^{R(x)}}.$$

Given

$$F(x) = a_1 x + a_2 x^2 + \cdots, \ a_1 \neq 0,$$

define $F(x)^{\boldsymbol{\langle -1 \rangle}}$ by

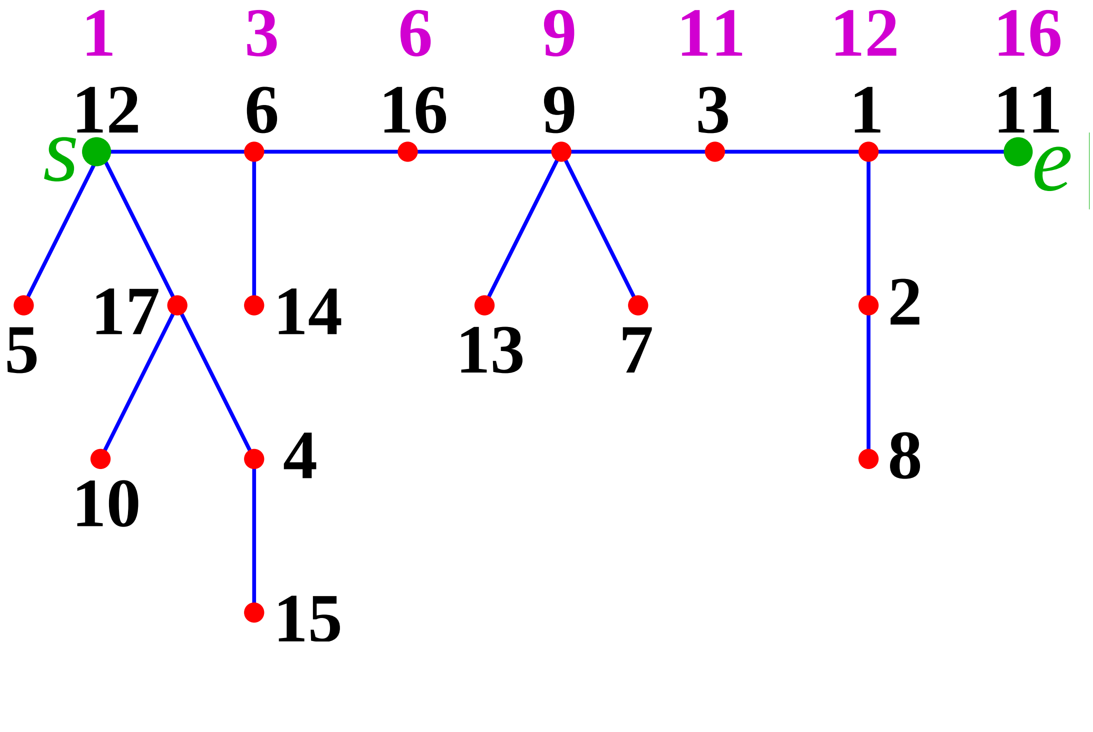
$$F(F^{\langle -1 \rangle}(x)) = F^{\langle -1 \rangle}(F(x)) = x$$

(exists and is unique). Then

$$\begin{aligned} R(x) &= xe^{R(x)} \\ \Rightarrow R(x) &= (xe^{-x})^{\langle -1 \rangle}. \end{aligned}$$

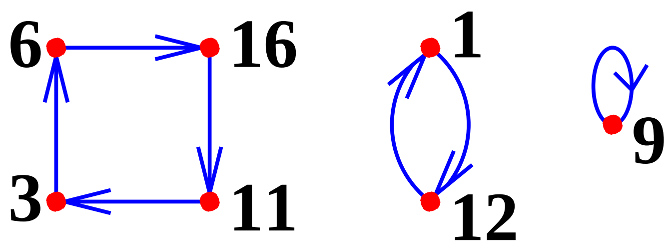
How to find the coefficients $r(n)/n!$ of $(xe^{-x})^{\langle -1 \rangle}$?

Bijective proof (Joyal). A **double rooted tree** is a tree with one vertex labelled s (start) and one vertex (possibly the same) labelled e (end). The number of double rooted trees on $[n]$ is $n \cdot r(n)$. Let $\boldsymbol{T}$ be such a tree, and let $\boldsymbol{P}$ be the unique path from s to e.

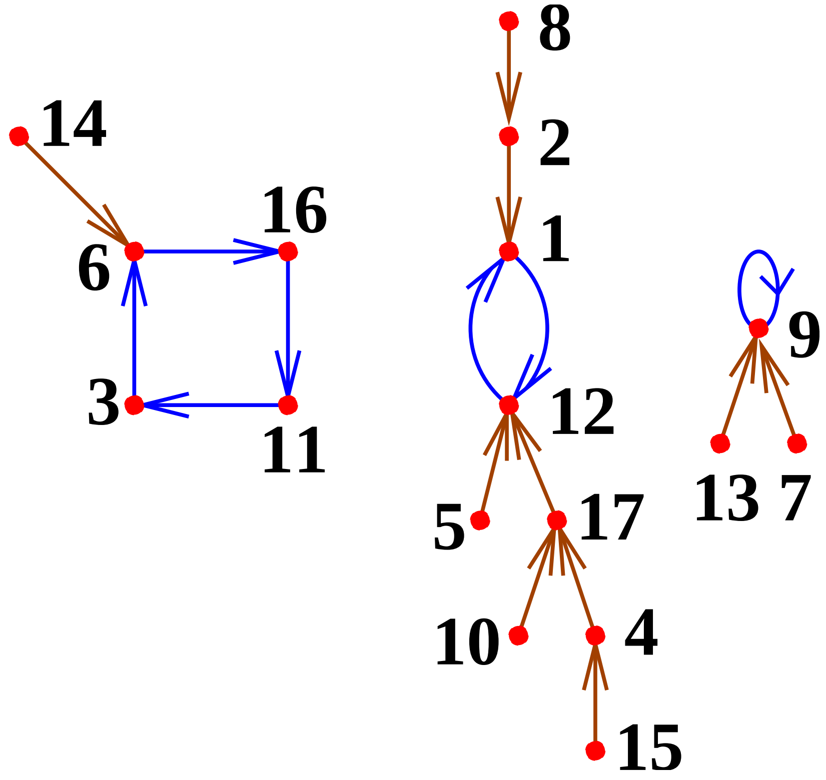


The vertices from s to e form a permutation of its elements written in increasing order. Write this permutation in cycle form as a directed graph:

1	**3**	**6**	**9**	**11**	**12**	**16**
12	**6**	**16**	**9**	**3**	**1**	**11**



Attach the subtrees of the path P back to their attached vertices and directed into the cycles:



We obtain a digraph on $[n]$ for which every vertex has outdegree one, i.e., the graph of a function $f : [n] \to [n]$. Conversely, every such f comes from a unique double rooted tree T.

$$\# \text{ of } f : [n] \to [n]\text{:} \quad \boldsymbol{n^n}$$
$$\Rightarrow \# \text{ double-rooted trees on } [n]\text{:} \quad \boldsymbol{n^n}$$
$$\Rightarrow r(n) = \boldsymbol{n^{n-1}}$$

Can we generalize this argument to find coefficients of other $F^{\langle -1 \rangle}(x)$?

Lagrange inversion formula. *Let*

$$\boldsymbol{F(x)} = a_1 x + a_2 x^2 + \cdots, \quad a_1 \neq 0.$$

Let $k, n \in \mathbb{Z}$. *Then*

$$n[x^n]F^{\langle -1 \rangle}(x)^k = k[x^{n-k}]\left(\frac{x}{F(x)}\right)^n.$$

Proof. A combinatorial proof can be given based on counting trees. Proof of Lagrange:

Consider **Laurent series**

$$G(x) = \sum_{n \geq n_0 \in \mathbb{Z}} b_n x^n.$$

For instance,

$$\begin{aligned} \frac{1}{F(x)^k} &= \frac{1}{(a_1 x + a_2 x^2 + \cdots)^k} \\ &= \frac{1}{x^k (a_1 + a_2 x + \cdots)^k} \\ &= x^{-k}(d_0 + d_1 x \cdots) \\ &= d_0 x^{-k} + d_1 x^{-k+1} + \cdots. \end{aligned}$$

Key fact:

$$[x^{-1}]\frac{d}{dx}G(x) = 0$$

Set $F^{\langle -1\rangle}(x)^k = \sum_{i\geq k} p_i x^i$, so

$$x^k = \sum_{i\geq k} p_i F(x)^i.$$

Apply $\frac{d}{dx}$:

$$kx^{k-1} = \sum_{i\geq k} i p_i F(x)^{i-1} F'(x)$$

$$\Rightarrow \frac{kx^{k-1}}{F(x)^n} = \sum_{i\geq k} i p_i F(x)^{i-n-1} F'(x).$$

Take $[x^{-1}]$ on both sides. Since

$$F(x)^{i-n-1}F'(x) = \frac{1}{i-n}\frac{d}{dx}F(x)^{i-n}, \;\; i \neq n,$$

the coefficient of x^{-1} of the right-hand side is

$$\begin{aligned}
[x^{-1}]np_n\frac{F'(x)}{F(x)} &= [x^{-1}]np_n\left(\frac{a_1 + 2a_2x + \cdots}{a_1x + a_2x^2 + \cdots}\right) \\
&= [x^{-1}]np_n\left(\frac{1}{x} + \cdots\right) \\
&= np_n.
\end{aligned}$$

Hence

$$[x^{-1}]\frac{kx^{k-1}}{F(x)^n} = np_n = n[x^n]F^{\langle -1\rangle}(x)^k,$$

which is equivalent to

$$n[x^n]F^{\langle -1\rangle}(x)^k = k[x^{n-k}]\left(\frac{x}{F(x)}\right)^n. \;\Box$$

Let

$$R(x) = (xe^{-x})^{\langle -1 \rangle} = \sum_{n \geq 1} r(n) \frac{x^n}{n!}.$$

Thus if $\boldsymbol{r_k(n)}$ is the number of forests of k rooted trees on $[n]$, then

$$\frac{1}{k!} R(x)^k = \sum_{n \geq k} r_k(n) \frac{x^n}{n!}.$$

By Lagrange inversion,

$$\begin{aligned} n[x^n]R(x)^k &= k[x^{n-k}]\left(\frac{x}{xe^{-x}}\right)^n \\ &= k[x^{n-k}]e^{nx} \\ &= \frac{kn^{n-k}}{(n-k)!}, \end{aligned}$$

so

$$\begin{aligned} r_k(n) &= \frac{k}{n}\frac{n!}{(n-k)!k!}n^{n-k} \\ &= \binom{n-1}{k-1}n^{n-k}. \end{aligned}$$

ALGEBRAIC FUNCTIONS

A power series $F(x) = a_0 + a_1 x + \cdots$ is **algebraic** if $\exists$ a polynomial $L(u, v) \neq 0$ such that

$$L(x, F(x)) = 0.$$

Examples. (a) Rational functions $F(x) = P(x)/Q(x)$ are algebraic, since

$$Q(x)F(x) - P(x) = 0.$$

(b) Easy to check that

$$\binom{-1/2}{n} = \left(-\frac{1}{4}\right)^n \binom{2n}{n},$$

so

$$F(x) := \sum_{n \geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}.$$

(c) Let $F(x) = \sum_{n\geq 0} \binom{3n}{n} x^n$. Then

$$(27x-4)F(x)^3 + 3F(x) + 1 = 0.$$

(d) **Not** algebraic:

$$\sum_{n\geq 0} \binom{2n}{n}^2 x^n, \quad \sum_{n\geq 0} \frac{(3n)!}{n!^3} x^n.$$

Theorem. Let $F(x) = \sum_{n\geq 0} f(n)x^n$ be algebraic. Then $\exists\, d \geq 1$ and polynomials $P_0(n), \ldots, P_d(n)$ (not all 0) such that for all $n \geq 0$

$$P_d(n)f(n+d) + P_{d-1}(n)f(n+d-1) + \cdots + P_0(n)f(n) = 0.$$

One says $F(x)$ is **D-finite** and $f(n)$ is **P-recursive**.

Proof (sketch). Let $L(u,v)$ be a nonzero polynomial such that $L(x,F(x))=0$. Thus

$$L_u(x,F(x))+F'(x)L_v(x,F(x))=0$$

$$\Rightarrow F'(x)=-\frac{L_u(x,F(x))}{L_v(x,F(x))}\in\mathbb{C}(x,F(x)).$$

Similarly all higher derivatives $F^{(i)}(x)\in\mathbb{C}(x,F(x))$. Since $F(x)$ is algebraic

$$\dim_{\mathbb{C}(x)}\mathbb{C}(x,F(x))<\infty.$$

Thus $F(x),F'(x),F''(x),\ldots$ are linearly dependent over $\mathbb{C}(x)$. Write down this linear dependence relation, clear denominators, and equate coefficients of x^n to get an equation

$$P_d(n)f(n+d)+\cdots+P_0(n)f(n)=0.\ \square$$

Example. Let **f(m,n)** be the number of paths from $(0,0)$ to (m,n) with steps $(1,0)$, $(0,1)$, $(1,1)$ (**Delannoy number**). Thus

$$\sum_{m,n\geq 0} f(m,n)x^m y^n = \sum_{k\geq 0}(x+y+xy)^k = \frac{1}{1-x-y-xy}.$$

Then

$$y := \sum_{n\geq 0} f(n,n)x^n = [t^0]\frac{1}{1 - xt - \frac{1}{t} - x}$$

$$= [t^0]\frac{1}{\beta-\alpha}\left(\frac{t}{t-\alpha} - \frac{t}{t-\beta}\right),$$

where $\alpha = \frac{1}{2}(1 - x - \sqrt{1-6x+x^2})$, $\beta = \frac{1}{2}(1 - x + \sqrt{1-6x+x^2})$. Hence

$$y = [t^0]\frac{1}{\sqrt{1-6x+x^2}}\left(\frac{t\alpha^{-1}}{1-t\alpha^{-1}} + \frac{1}{1-t^{-1}\beta}\right)$$

$$= \frac{1}{\sqrt{1-6x+x^2}},$$

and we get for $g(n) = f(n,n)$,

$$(n+2)g(n+2) - 3(2n+3)g(n+1) + (n+1)g(n) = 0$$

(challenging to prove directly!).

k-ARY PLANE TREES

A **k-ary plane tree** is a rooted tree for which every non-endpoint vertex has k cyclically ordered subtrees.

Let $\boldsymbol{f_k(n)}$ denote the number of k-ary plane trees with n vertices and

$$y = \boldsymbol{F_k(x)} = \sum_{n\geq 0} f_k(n)x^n.$$

Then $y = x + xy^k$, so

$$y = \left(\frac{x}{1+x^k}\right)^{\langle -1\rangle}.$$

By Lagrange inversion,

$$n[x^n]y = [x^{n-1}](1+x^k)^n$$

$$\Rightarrow f_k(n) = \begin{cases} \frac{1}{n}\binom{n}{j}, & n = kj+1 \\ 0, & \text{otherwise.} \end{cases}$$

Special case: $k = 2$ (plane **binary** trees). Then

$$f_2(2n+1) = \frac{1}{n+1}\binom{2n}{n},$$

a **Catalan number** $\boldsymbol{C_n}$.

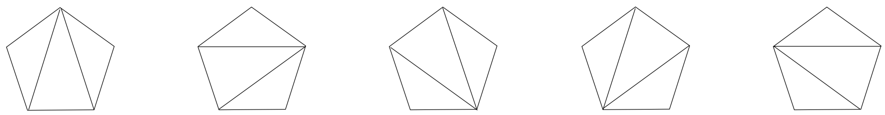
66 combinatorial interpretations of C_n: Exercise 6.19 of *Enumerative Combinatorics*, vol. 2.

36 additional interpretations (as of 22 December 2002):

`www-math.mit.edu/~rstan/ec`

Examples.

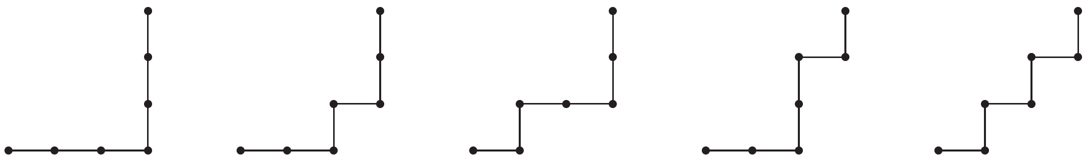
- triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors



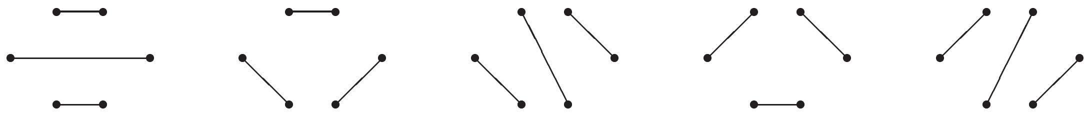
- binary parenthesizations of a string of $n+1$ letters

$$(xx{\cdot}x)x \quad x(xx{\cdot}x) \quad (x{\cdot}xx)x \quad x(x{\cdot}xx) \quad xx{\cdot}xx$$

- lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ or $(1,0)$, never rising above the line $y=x$



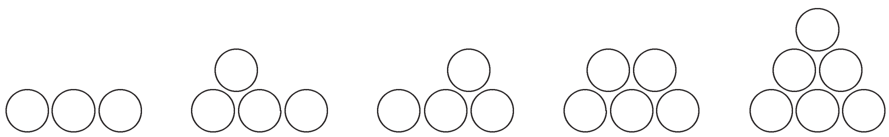
- n nonintersecting chords joining $2n$ points on the circumference of a circle



- permutations $a_1 a_2 \cdots a_n$ of $[n]$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$)

123 213 132 312 231

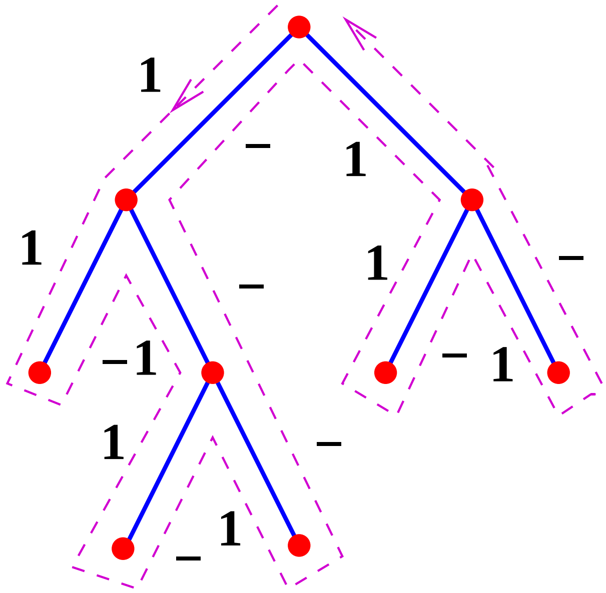
- ways to stack coins in the plane, the bottom row consisting of n consecutive coins



- n-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1a_1a_2 \cdots a_n1$, each a_i divides the sum of its two neighbors

  14321 13521 13231 12531 12341

Bijective proof that there are $C_n = \frac{1}{n+1}\binom{2n}{n}$ plane binary trees with $2\,n+1$ vertices: do a depth-first (preorder) search through the tree, labeling down edges 1, up edges -1, and ignoring the last edge.



1 1 – 1 1 – 1 – – – 1 1

This converts trees to sequences of $n+1$ 1's and n -1's such that every partial sum is positive.

Claim. For any sequence $a_1a_2\cdots a_{2n+1}$ of $n+1$ 1's and n -1's, there is exactly one value of i for which every partial sum of $a_ia_{i+1}\cdots a_{2n+1}a_1a_2\cdots a_{i-1}$ is positive.

Claim. For any sequence $a_1 a_2 \cdots a_{2n+1}$ of $n+1$ 1's and n -1's, there is exactly one value of i for which every partial sum of $a_i a_{i+1} \cdots a_{2n+1} a_1 a_2 \cdots a_{i-1}$ is positive.

Proof (naive). Induction on n. Clear for $n = 0$. Assume for $n-1$. Given $\alpha = a_1 \cdots a_{2n+1}$, can always find $a_j = 1$, $a_{j+1} = -1$ (subscripts modulo $2n+1$). Remove a_j, a_{j+1} from α, giving $\beta = b_1 \cdots b_{2n-1}$. By the induction hypothesis there is a unique i for which $b_i \cdots b_{i-1}$ has all partial sums positive. If $b_i = a_k$, then k is the unique integer for which $a_k \cdots a_{k-1}$ has every partial sum positive. $\square$

There are $\binom{2n+1}{n}$ sequences of $n+1$ 1's and n -1's. All their $2n+1$ "cyclic shifts" are distinct since $\gcd(n, n+1) = 1$. Thus the number of plane binary trees with $2n+1$ vertices is

$$\frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n} = C_n.$$