Symmetric Venn Diagrams on the Sphere I: Chain Decompositions and Isometry Groups of Order Two

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September 24, 2009

Abstract

Let f be an isometry of the sphere which is an involution (that is, $f^{-1} = f$). For any $n \geq 1$ we prove that there is an n-curve Venn diagram on the sphere whose isometry group is generated by f and which has the property that $f(\Theta) = \Theta$, where Θ is a curve of the diagram. One of the constructions uses a new chain decomposition of the Boolean lattice, a decomposition which is based on the well-known decomposition of N. G. de Bruijn, C. A. van Ebbenhorst Tengbergen, and D. R. Kruyswijk.

1 Introduction

The construction of rotationally symmetric Venn diagrams has garnered considerable interest in the last decade, due in part to accessible open problems, ties with other parts of mathematics, and the intrinsic beauty of renderings of the actual diagrams. It is natural to consider Venn diagrams drawn on the sphere and to hope that the richer set of possible symmetries would lead to additional classes of Venn diagrams which realize other interesting groups of isometries. In this paper we consider isometries of order two and where each curve gets mapped to itself. In successor papers ([16],[17]), we will consider other classes of isometries.

1.1 Venn Diagrams in the Plane

If a curve in the plane is closed and simple, then the Jordan Curve Theorem tells us that the curve separates the plane into a bounded interior and an unbounded exterior. Following

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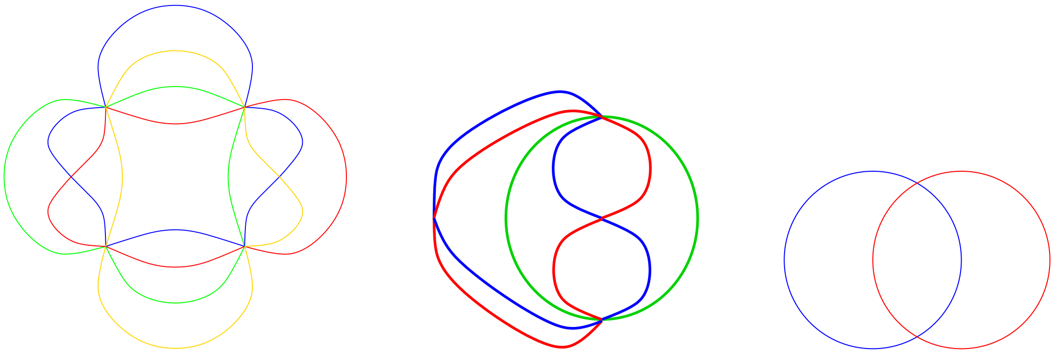
†Research supported in part by NSERC.



(a) Monochrome symmetric but not colour symmetric

(b) Colour symmetric but not totally symmetric

(c) Totally symmetric

Figure 1: Three Venn diagrams with different kinds of colour symmetries

Grünbaum [10], a n-curve *Venn diagram* is defined to be a collection of n simple closed curves, $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, with the property that for each $S \subseteq \{1, 2, \ldots, n\}$ the *region*

$$\bigcap_{i \in S} \text{interior}(\theta_i) \quad \cap \quad \bigcap_{i \notin S} \text{exterior}(\theta_i) \tag{1}$$

is nonempty and connected. In a Venn diagram the curves are assumed to intersect in only finitely many points. An n-*Venn diagram* is one with n curves. A Venn diagram is *simple* if no three curves have a common point of intersection.

We say that Θ is *monochrome symmetric* if there is a non-trivial isometry of the plane, f, such that

$$f(\theta_1 \cup \theta_2 \cup \cdots \cup \theta_n) = \theta_1 \cup \theta_2 \cup \cdots \cup \theta_n.$$

If there is a permutation σ of $\{1, 2, \ldots, n\}$ such that

$$f(\theta_j) = \theta_{\sigma(j)} \text{ for each } 1 \leq j \leq n,$$

then Θ is *colour symmetric*. Finally, if the permutation σ is the identity then Θ is said to be *totally symmetric*. These definitions are illustrated in Figure 1.

For each of the three symmetry types, every Venn diagram will have a group of isometries. Now the possible isometries of the plane are well-known (see for example [2]): they are all composed of subsets of (a) translations, (b) reflections, or (c) rotations. Translations are not of interest to us because no bounded subset of $\mathbb{R}^2$ is fixed by a translation.

Most research on symmetric Venn diagrams has been concerned with colour symmetry where the isometry is a rotation of the plane; see the survey [14] for discussions and examples regarding diagrams with this type of symmetry. A recent result by Griggs, Killian and Savage [9] that we use in this work provides a construction for a colour symmetric n-Venn diagram for any prime number n; here the isometry group consists of rotations of the plane by $2\pi/n$. This construction will be modified in later sections to create symmetric Venn diagrams on the sphere.

1.2 The Dual of a Venn Diagram

A simple n-Venn diagram Θ may be regarded as a planar graph, where the vertices are the intersection points, and the portions of the curves between vertices are edges. Since Θ is a planar graph, we may form the dual graph, denoted $D(\Theta)$. It is known that $D(\Theta)$ is a planar spanning subgraph of the hypercube Q_n. We will regard the vertices of the hypercube Q_n as being labelled by the set of 2^n bitstrings of length n. If $b_1 b_2 \cdots b_n$ is a generic vertex, then $b_i = 1$ indicates that the region represented by this vertex is in the interior of curve i; if $b_i = 0$, then it is in the exterior. Now consider an isometry, f, of a n-Venn diagram Θ and a point x not on Θ. We may associate a bitstring $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ with x according to which curves include it or not, as in the previous sentence. If f is a colour symmetry then it induces a permutation π of the curves, and correspondingly permutes the bits $\sigma_1 \sigma_2 \cdots \sigma_n$ of σ to obtain the bitstring $\sigma_{\pi(1)} \sigma_{\pi(2)} \cdots \sigma_{\pi(n)}$. This correspondence will be exploited in future sections when we work primarily with the dual graphs of Venn diagrams.

1.3 The Total Symmetries of Planar Venn Diagrams

Total symmetries are very restrictive, particularly so in the plane. We begin with a simple technical lemma.

Lemma 1.1. *If f is a total symmetry of a Venn diagram, then each region R of the diagram is invariant under f; i.e., $f(R) = R$.*

Proof. Since for any curve θ of the diagram $f(\theta) = \theta$, it follows that $f(\text{int}(\theta)) = \text{int}(\theta)$ and $f(\text{ext}(\theta)) = \text{ext}(\theta)$. Also, for sets $A, B \subseteq \mathbb{R}^2$ it is clear that $f(A \cap B) = f(A) \cap f(B)$. Thus if $R = (\bigcap_{i \in S} \text{int}(\theta_i)) \cap (\bigcap_{i \notin S} \text{ext}(\theta_i))$ then

$$f(R) = \bigcap_{i \in S} f(\text{interior}(\theta_i)) \quad \cap \quad \bigcap_{i \notin S} f(\text{exterior}(\theta_i)) = R$$

□

Theorem 1.2. *If f is a non-trivial total symmetry of an n-Venn diagram Θ, then either*
(a) f is a reflection and $n = 1$ or $n = 2$; or
(b) f is a rotation, or a rotation followed by a reflection, and $n = 1$.

Proof. If θ is a curve invariant under the reflective isometry f, then θ must intersect the reflective line ℓ at exactly two points. If Θ is a Venn diagram, then each of its regions R must have non-empty intersection with ℓ (otherwise R and $f(R)$ are disconnected, but we must have $f(R) = R$). Now let θ_1, θ_2, and θ_3 be three curves of Θ. The two curves θ_1 and θ_2 induce four regions, which we denote $R_\emptyset$, R_1, R_2 and R_{12}. The curve θ_3 must intersect each of these regions. Now consider a point x on θ_3. As we follow θ_3 clockwise it will eventually intersect ℓ or one of the other two curves; the same is true if we follow it counter-clockwise. We claim that in one of those directions it will will first hit ℓ. For, if not, then it would create a region that does not intersect ℓ. But since there are four regions, we conclude that θ_3 intersects ℓ in at least four places, which is a contradiction. Thus θ_3 does not exist; i.e., $n = 1$ or $n = 2$.

Suppose that a Venn diagram Θ has the rotation ρ as a symmetry, where the rotation angle is $2\pi/m$ and the fixed point of the rotation is r.

We first argue that no curve θ of Θ contains r. For if it did then consider a point x of $\text{int}(\theta)$ that is a very small distance from θ. After the rotation we have $x \in \text{ext}(\theta)$, which is a contradiction to the observation made in the proof of the lemma above. Thus r does not lie on any curve.

Let y be a point in the interior of some region R, different than the region containing r. Since R is connected and $\rho(y) \in R$, there must be a curve whose endpoints are y and $\rho(y)$, and that is contained in R. But there would be a congruent curve connecting $\rho^k(y)$ and ρ^{k+1} for any natural number $k = 1, 2, \ldots, m$. The union of all m of those congruent curves is a simple closed curve, call it ξ.

The curve ξ partitions the curves of Θ into two disjoint sets, those contained in $\text{int}(\xi_i)$ and those contained in $\text{ext}(\xi_i)$. Obviously, in a Venn diagram one of those disjoint sets must be empty, and since y was chosen not to be in the region containing r, ξ must lie in the unbounded exterior of the Venn diagram. Thus y must lie in that unbounded exterior. We conclude that the Venn diagram partitions the plane into two regions, and this only happens if $n = 1$. $\square$

1.4 The Sphere and its Isometries

It is natural to consider Venn diagrams drawn on the sphere instead of on the plane, and to hope that the richer set of isometries of the sphere will lead to new types of symmetric diagrams.

Grünbaum was the first to discuss Venn diagrams on the sphere [11, 12], though only in the context of identifying when two diagrams are isomorphic on the sphere. We discuss the symmetries of the sphere before considering diagrams on the sphere.

The *2-sphere* (or *sphere*), is the surface S in $\mathbb{R}^3$ given by $S = \{(x, y, z) \in \mathbb{R}^3 | x^2+y^2+z^2 = r^2\}$ for some fixed radius r; we choose $r = 1$ for convenience. We can identify points on the sphere by Cartesian coordinates in three dimensions, or spherical coordinates. Assuming a unit radius sphere, in spherical coordinates a point p on the sphere can be located by two coordinates (θ, ϕ), where ϕ is the angle between the equatorial plane and the line containing the origin and p, and θ is the angle between the positive x-axis and the line containing the origin and p projected onto the equatorial plane.

In Euclidean space all isometries can be classed into six types: *reflection* across a plane, *rotation* about an axis, *translation* along a vector, *screw rotation* (rotation followed by translation in a direction along the rotational axis), *rotary reflection* (rotation followed by a reflection across a plane orthogonal to the rotational axis), and *glide reflection* (translation followed by reflection across a plane containing the translation vector) [2]. An isometry of the sphere must fix its centre, so all isometries of the sphere are types of reflection, rotation, and rotary reflection. The set of all isometries of the sphere is usually written $O(3)$. In this work we are only interested in isometries of order two: that is, mappings $f \in O(3)$ so that for any point $p \in S$, we have $f(f(p)) = p$ and $f(p) \neq p$. There are three such isometries:

rotation by π: Denote this isometry by R. Without loss of generality we may assume that the z axis is the axis of rotation. In spherical coordinates, $R(\phi, \rho) = (\phi, \rho + \pi)$.

reflection about a plane: Denote this isometry by F. Without loss of generality we may assume that the $x = 0$ is the plane of reflection. In spherical coordinates, $F(\phi, \rho) = (-\phi, \rho)$.

inversion or antipodal symmetry: Denote this isometry by G (due to its similarity to glide reflection on the plane). Inversion is a specific type of rotary reflection, and is obtained by composing R with F (or vice-versa as the two mappings commute). In spherical coordinates, $G(\phi, \rho) = (-\phi, \rho + \pi)$.

On the plane the interior and exterior of a simple closed curve are well-defined since only the exterior is unbounded, but on the sphere each curve partitions the surface into two bounded parts, and we are free to choose either part as the interior of a curve. An n-Venn diagram on the sphere is defined the same way as it was in the plane, except that in Equation 1, interior(θ_i) and exterior(θ_i) denote the arbitrary choices that we have made about which part of the sphere are the bounded interior and the bounded exterior, respectively, of θ_i.

In our figures we use cylindrical projections of spherical diagrams onto the plane; a spherical figure is "unwrapped" by equating lines of latitude (parallel to the equator) with parallel horizontal lines, and vertical lines of longitude map to parallel vertical lines: each pole maps to a horizontal bounding line of the projection.

1.5 The Boolean Lattice and Chain Decompositions

We assume some basic familiarity with posets. In this work we will be dealing with the poset called the *Boolean* (or *subset*) *lattice of order* n. In the Boolean lattice, the elements are the set B_n, corresponding to the 2^n subsets of the n-set, and they are related by subset inclusion; the Boolean lattice is written $\mathcal{B}_n$, where $\mathcal{B}_n = (B_n, \subseteq)$. These subsets correspond naturally to the 2^n bitstrings of length n where a bit in position i in a bitstring is 1 if and only if the subset includes element i. Two bitstrings a, b are related (and we also say $a \prec b$) if and only if their corresponding subsets are related such that $a \subset b$. Thus, the cover relations are exactly those pairs of bitstrings a, b such that a and b differ in one bit position where a has a 0 and b has a 1. In the Hasse diagram of the Boolean lattice, the all-zero bitstring 0^n, corresponding to the empty set, appears as the lowest element, and the all-one bitstring 1^n as the highest.

A *chain* in a poset is a set of elements, each pair of which is comparable. A chain is *maximal* if no element can be added to it. Furthermore, a chain C in a poset P is called *saturated* if there does not exist $b \in P - C$ such that $a \prec b \prec c$ for some $a, c \in C$, and such that $C \cup \{b\}$ is a chain. All chains are saturated unless otherwise specified. We will always write a chain's elements in the order $\{\ \sigma_1,\ \sigma_2,\ \ldots\ , \sigma_k\ \}$, where $\sigma_i\ \prec\ \sigma_j$ for $i < j$, and the braces can be omitted for brevity.

The poset $\mathcal{P}$ is *ranked* if all maximal chains have the same length; in the case of the Boolean lattice all maximal chains have length $n + 1$, which is the poset's *height*. Thus, the poset can be partitioned into ranks, where every maximal chain consists of exactly one element from each rank. The ranks can be indexed by their distance (in the graph-theoretic sense) in the Hasse diagram from an element with the lowest height; in the Boolean lattice the natural rank for a bitstring is its weight, which is its distance from the extremal element 0^n.

A *chain decomposition* is a partition of a poset into chains. A chain $C = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ in a ranked poset P of height h is called a *symmetric* chain if there exists an integer i such that C contains exactly one element from each rank $A_i, A_{i+1}, \ldots, A_{h+1-i}$. A symmetric chain is thus "balanced" about the middle rank (or middle two ranks, if the height is even) of the poset and saturated. A *symmetric chain decomposition* is a partition of a poset into symmetric saturated chains; for the Boolean lattice there are thus $\binom{n}{\lfloor n/2 \rfloor}$ chains in a symmetric chain decomposition.

The Hasse diagram of the Boolean lattice $\mathcal{B}_n$, when viewed as a graph, is isomorphic to the n-cube Q_n. As noted earlier the isomorphism maps a set $S \subseteq \{1, \ldots, n\}$ to the n-bit string with a 1 in position i if and only if $i \in S$. The *weight* of a bit string is the number of ones it contains.

A very nice chain decomposition for the Boolean lattice $\mathcal{B}_n$ was given by de Bruijn, et al. in 1951 [3], and studied by many subsequent authors, including the most well-used presentation by Greene and Kleitman [8].

Knuth [13] presents the de Bruijn chain decomposition as a recursive construction as follows. The base case is the decomposition for $n = 1$, which is the single chain $\{0, 1\}$. Given a chain decomposition of the order-n lattice, to create the chain decomposition for the order-$(n+1)$ lattice, take every chain $\{\sigma_k, \sigma_{k+1}, \ldots, \sigma_{k+j}\}$, where σ_i is a bitstring of weight i, and replace it by the two chains

$$\begin{gathered}\{\ \sigma_{k+1}\, 0,\ \ldots\ ,\ \sigma_{k+j}\, 0\ \},\quad \text{and}\\ \{\ \sigma_k\, 0,\ \sigma_k\, 1,\ \ldots\ ,\ \sigma_{k+j-1}\, 1,\ \sigma_{k+j}\, 1\ \},\end{gathered}$$

and the first of these chains is omitted when $j = 0$ (so there is only one element in the chain in the order-n lattice).

In the important paper by Griggs, Killian and Savage [9], it is shown how this symmetric chain decomposition of $\mathcal{B}_n$ gives a straightforward construction for Venn diagrams for n curves. We refer interested readers to the original paper [9], or the survey [15], to fill in the details.

The results of this paper are, for each of the order-two symmetries (involutions) mentioned earlier, to create constructions for any n of an n-Venn diagram on the sphere that have the symmetry required. Of the four involutions, realizing an n-Venn diagram on the sphere with the identity symmetry is trivial, as any of the known constructions on the plane (see, for example, [21] or [6] or the survey [14]) can be embedded on the sphere. Constructions for realizing the first two non-trivial involutions stem from investigating some different symmetries created by different variations of the de Bruijn chain decomposition and ways of embedding the chains; our constructions are very similar to the Griggs, Killian, and Savage constructions. The third kind of symmetry results from expanding on previous discussions of a known construction by Edwards.

2 Symmetries in Posets

In this section we introduce some further terminology and concepts regarding posets that we exploit later for the purposes of building Venn diagrams. We follow Trotter [19, 20] in our definitions.

When $P = (X, R)$ and $Q = (Y, S)$ are posets, a map $f : X \to Y$ is *order-preserving* (respectively, *order-reversing*) if $x_1 \prec x_2$ in P implies $f(x_1) \prec f(x_2)$ (respectively, $f(x_1) \succ f(x_2)$) in S for all $x_1, x_2 \in X$. Furthermore, if two posets have an order-preserving (respectively, order-reversing) bijection between them, they are *isomorphic* (respectively, *dual-isomorphic*). An isomorphism from P to P is called an *automorphism*, and a dual-isomorphism from P to P is called a *dual-automorphism*. A poset with a dual-automorphism is termed *self-dual*.

Many of the classic posets studied in combinatorics are self-dual, including the Boolean lattice. The *dual* of a poset $P = (X, R)$ is the poset $P^* = (X, R^*)$ where $x \prec y$ in P if and only if $y \prec x$ in P^*; the Hasse diagram of P^* is the Hasse diagram of P drawn upside down.

We can extend these notions to chain decompositions, as some dual-automorphisms preserve the chains in a given chain decomposition. Given two posets P and Q and chain decompositions C of P and E of Q, a *chain decomposition isomorphism* (respectively, *chain decomposition dual-isomorphism*) is an order-preserving (respectively, order-reversing) map $f : P \to Q$ such that $f(C_i) = E_j$ for some $C_i \in C$ and $E_j \in E$.

A chain decomposition isomorphism (respectively, dual-isomorphism) from a poset and chain decomposition P, C to P, C is a *chain decomposition automorphism* (respectively, *chain decomposition dual-automorphism*). Again, a chain decomposition that has a chain decomposition dual-automorphism is called *self-dual*.

For example, in Figure 3, let $f(x_1x_2x_3x_4) = \overline{x_1}\,\overline{x_2}\,\overline{x_3}\,\overline{x_4}$. Then f is a chain decomposition dual-automorphism on this chain decomposition of $\mathcal{B}_4$. Consider numbering the chains from top to bottom $1 \ldots 6$, then f also has the property that $f(C_i) = C_{7-i}$.

For another example, Knuth showed in [13] that the de Bruijn chain decomposition of $\mathcal{B}_n$ is self-dual under the dual-automorphism $\phi : \phi(x) = \overline{x}^R$.

2.1 Embeddings of Chain Decompositions

Some chain decompositions have the property that they form part of the dual graph of a Venn diagram on the plane; for this to be true there must be a planar embedding of the chains on the plane with appropriate additional edges so that they are linked into a connected graph. In the construction of [9], chains in the symmetric chain decomposition of $\mathcal{B}_n$ are laid out on the plane in a specific order so that, once the chains are connected by chain cover edges and the dual graph taken, the dual forms a n-Venn diagram.

We begin by slightly modifying the appropriate definitions from the paper [9]. The concept of the chain cover property applying to a chain decomposition can be broadened to include non-symmetric chains, and allow chains to be covered by two different chains (one at each end).

A chain decomposition $\mathcal{C}$ has the *chain cover property* if whenever $C \in \mathcal{C}$, then there exists two chains $\pi(C), \pi'(C) \in \mathcal{C}$ such that *starter*(C) covers a starter element $\pi_s(C)$ of $\pi(C)$, and

terminator(C) is covered by an element $\pi'_t(C)$ of $\pi'(C)$,

unless *starter*(C) is of the smallest rank, in which case $\pi(C)$ does not exist, or *terminator*(C) is of the largest rank, in which case $\pi'(C)$ does not exist. Again π is called the *chain cover mapping*.

For example, in Figure 3 the chain $\{0000, 0001, 0011, 0111\}$ has no chain covered by its starter 0000, since 0000 is of minimal rank, but the terminator 0111 is covered by 1111, from the neighbouring chain.

In [9] the concept of a chain decomposition is developed along with its embedding in the plane, via the concept of the chain cover graph. In this section we supersede this concept, since the use of the chain cover graph in [9] to embed the entire chain decomposition and all of its cover edges depend on the chain decomposition being symmetric, which is not the case for the decompositions in this section.

In [9] the notion is developed that, given the chain decomposition $\mathcal{C}$ with the chain cover property, we can define the *chain cover graph* $G(\mathcal{C}, \pi)$. The vertices of $G(\mathcal{C}, \pi)$ are the elements of the poset that $\mathcal{C}$ covers, and the edges consist of the covering relations in the chains in $\mathcal{C}$ together with the *cover edges*, for each chain C, from *starter*(C) to $\pi_s(C)$ and from *terminator*(C) to $\pi'_t(C)$.

For a given n the constructions from the paper [9] have only one embedding and so no particular way of differentiating between different embeddings was required. To overcome this, we define a *chain decomposition embedding* of a chain decomposition $\mathcal{C}$ with the chain cover property as a planar embedding of the chain cover graph $G(\mathcal{G}, \pi)$ in which the chains are laid out sequentially in parallel, with all elements of the same rank in a row.

Such a chain decomposition embedding specifies the order of chains to be laid out in the plane (and thus the positioning of the cover edges); thus it makes sense, given such an embedding, to refer to the first chain C_1, second chain C_2, and so on, where the first chain is identified as one of the chains that is extremal along the axis the chains are laid out on. For example, Figure 3 shows a chain decomposition embedding of $\mathcal{B}_4$ with each chain embedded in a horizontal row and chains progressing vertically from C_1 uppermost to C_6 at the bottom. A chain embedded in a chain decomposition embedding can be referred to as a *row* of the chain decomposition embedding.

Without loss of generality, we will always index chains in an embedding from top to bottom or left to right (depending on the particular layout of the chains, whether horizontal or vertical).

3 Reverse Symmetric Chain Decomposition Embeddings

In this section we discuss a variation of the de Bruijn decomposition of the Boolean lattice used in [9] to create monotone Venn diagrams. This variation proves to have a more interesting symmetry on the sphere. We define the construction recursively, following Knuth [13, p. 17].

The notions of chain decomposition isomorphisms and dual-isomorphisms from the previous sections can also be applied to chain decomposition embeddings. Let E be a chain decomposition embedding of $\mathcal{B}_n$ with k chains, with chains numbered sequentially (as described in Section 2.1). Then E is *reverse symmetric* if the function $f(C_i) = C_{k+1-i}$ reversing the order of the chains, and applying $\phi(x) = \overline{x}$ to the elements is a dual-automorphism of the embedding.

Thus, the chain decomposition maps onto itself; *i.e* given a chain decomposition with m chains, the ith chain maps onto the $(m-i+1)$th chain in the decomposition and each element σ in the ith chain maps onto its complement $\overline{\sigma}$ in the $(m-i+1)$th chain. As an example of a reverse symmetric chain decomposition, in Figure 3, the first chain, $\{0100, 0101\}$, when complemented and reversed, gives $\{1010, 1011\}$, which is the last chain; the second chain, $\{0010, 0110\}$, when complemented and reversed, gives $\{1001, 1101\}$, the second last chain, and so on.

We modify the basis and recursive rule used to create the de Bruijn decomposition pattern to create a reverse symmetric chain decomposition embedding (RSCD), by proving that, as in the de Bruijn pattern, the RSCDs we construct have the chain cover property and planar embeddings of their chain cover graphs.

Basis. The basis for a recursive construction is a RSCD for $\mathcal{B}_2$, shown in Figure 2.

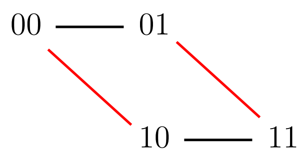


Figure 2: RSCD for $\mathcal{B}_2$, with chain cover edges shown in red

The chain cover edges in Figure 2 are the edges shown in red that connect different chains together; cover edges in the chains are shown in black.

The following recursive rules allow us to construct, given a RSCD for $\mathcal{B}_n$, a RSCD for $\mathcal{B}_{(n+1)}$.

Recursive Rule. Given a RSCD for $\mathcal{B}_n$ with m chains, the ith chain '$\sigma_1\ \sigma_2\ \ldots\ \sigma_k$', embedded as a row, is replaced in one of two ways, depending on whether the row is in the first half of the RSCD (*i.e.* whether $i \leq \frac{m}{2}$).

Rule 1. If $i \leq \frac{m}{2}$, the row becomes the two rows

$$\begin{array}{ccccc} & \sigma_2 0 & \ldots & \sigma_k 0 & \\ \sigma_1 0 & \sigma_1 1 & \ldots & \sigma_{k-1} 1 & \sigma_k 1, \end{array}$$

where if $k = 1$ the first row is omitted and the second becomes $\sigma_1 0\ \sigma_1 1$;

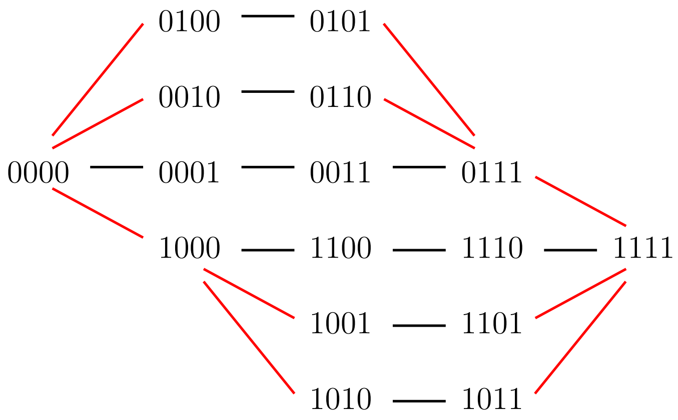


Figure 3: The RSCD for $\mathcal{B}_4$ generated by the recursive construction

Rule 2. if $i > \frac{m}{2}$, the row becomes

$$\begin{array}{ccccc} \sigma_1 0 & \sigma_2 0 & \dots & \sigma_k 0 & \sigma_k 1 \\ & \sigma_1 1 & \dots & \sigma_{k-1} 1, & \end{array}$$

where if $k = 1$ the second row is omitted and the first becomes $\sigma_k 0\ \sigma_k 1$.

Knuth notes in [13] (when discussing the de Bruijn construction), and thus we note also, that it is easy to show by induction that each of the 2^n bitstrings appears exactly once in the pattern, the bit strings with k 1s all appear in the same column, and within each row, consecutive bit strings differ by changing a 0 to a 1. The proof of each property has two cases for a string in C_i: one case for $i \leq \frac{m}{2}$ and one for $i > \frac{m}{2}$, and note that a string in the first half remains in the first half after application of the recursive rule, and similarly for the second half. The first two properties imply that there are $\binom{n}{k}$ elements in each column.

Observation 3.1. *Given a RSCD M for $\mathcal{B}_n$ and applying the construction to give a RSCD M' for $\mathcal{B}_{(n+1)}$, chains in M become chains in M'.*

Proof. By definition of Rules 1 and 2. □

Lemma 3.2. *Given a RSCD M for $\mathcal{B}_n$ and applying the construction to give a RSCD M' for $\mathcal{B}_{(n+1)}$, if M has reverse symmetry, M' does also.*

Proof. Rules 1 and 2 locally replace rows, so since there is no global replacement it suffices to show that if a row C_i maps onto a row in M then the same occurs in M', and the global property of being reverse is preserved.

Given a row $\sigma = \sigma_1\ \sigma_2\ \dots\ \sigma_k$ in the first $m/2$ rows of M, it maps onto its conjugate row $\omega = \omega_1\ \omega_2\ \dots\ \omega_k$ in the second $m/2$ rows of M; *i.e.* $\overline{\sigma_1} = \omega_k$, $\overline{\sigma_2} = \omega_{k-1}$, etc. Applying Rule 1 to σ gives the row

$$\begin{array}{ccccc} & \sigma_2 0 & \dots & \sigma_k 0 & \\ \sigma_1 0 & \sigma_1 1 & \dots & \sigma_{k-1} 1 & \sigma_k 1. \end{array} \tag{2}$$

Since ω is in the second half of M Rule 2 is applied, which gives

$$\begin{array}{ccccc} \omega_1 0 & \omega_2 0 & \dots & \omega_k 0 & \omega_k 1 \\ & \omega_1 1 & \dots & \omega_{k-1} 1. & \end{array} \tag{3}$$

It is easy to verify that the chains in (2) are the complements of those in (3), and since their order is reversed all chains in the first half of M' will map onto their complementary chains in the second half of M', as desired. □

We can ignore the embedding to obtain the following lemma.

Lemma 3.3. *The chain decomposition of $\mathcal{B}_n$ given by the RSCD construction of Rules 1 and 2 is self-dual under the bijection $\phi : \phi(x) = \overline{x}$.*

Proof. By Lemma 3.2. □

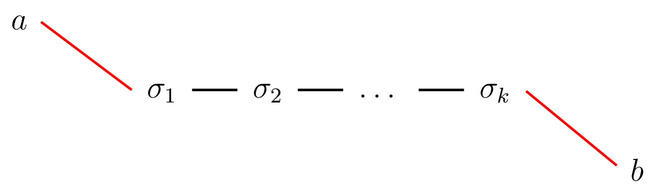
We now prove by induction that the RSCD for the n-cube has the chain cover property; the base case was shown in Figure 2.

Lemma 3.4. *Given a RSCD M for $\mathcal{B}_n$ and applying the construction to give a RSCD M' for $\mathcal{B}_{(n+1)}$, if M has the chain cover property, then the RSCD M' formed by applying Rules 1 and 2 also has the chain cover property.*

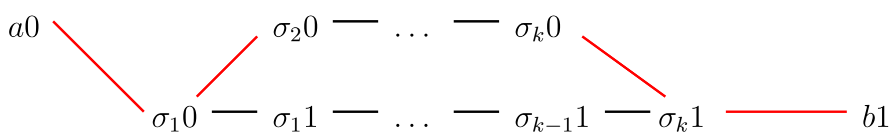
Proof. Since M has the chain cover property it suffices to show that the operation of replacing all rows according to Rules 1 and 2 creates new rows satisfying the chain cover property, and the chain cover edges will not introduce any edge crossings since the replacement rows and cover edges will be locally crossing-free.

Given a row $\sigma_1\ \sigma_2\ \dots\ \sigma_k$ in M, there are nodes a and b such that a covers σ_1 and σ_k covers b (note that a and b do not have to be part of the same chain), and a and b are the starter and terminator of their respective chains. After applying rules 1 and 2, a becomes $a0$ at the start of a longer chain (regardless of which rule is applied), and b will become $b1$ at the end of a longer chain.

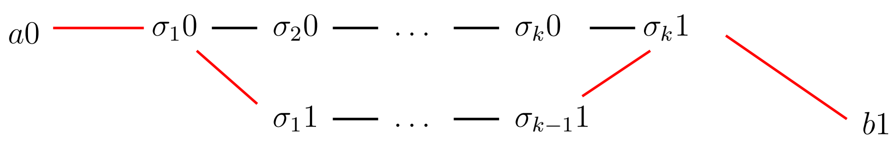
Given the chain



Rule 1 gives the chains:



and Rules 2 gives the chains:



If Rule 1 is applied, $a0$ covers $\sigma_1 0$ since a covered σ_1, $\sigma_k 1$ covers b_1 since σ_k covered b, $\sigma_1 0$ covers $\sigma_2 0$ since they differ by a single bit, and $\sigma_k 0$ covers $\sigma_k 1$ since they differ by a single bit.

If Rule 2 is applied, again $a0$ covers $\sigma_1 0$ and $\sigma_k 1$ covers b_1, $\sigma_1 0$ covers $\sigma_1 1$ since they differ by a single bit, and $\sigma_{k-1} 1$ covers $\sigma_k 1$ since they differ by a single bit.

None of the new edges introduce a crossing and so applying the expansion to all rows in M gives a RSCD M' with the chain cover property. □

Theorem 3.5. *There exists a reverse symmetric chain decomposition embedding of $\mathcal{B}_n$, for all $n \geq 2$, such that*

- *the number of chains is $N(n)$, as calculated in the next section in Theorem 3.9, and*
- *the chain decomposition has the chain cover property.*

Proof. The basis in Figure 2 is reverse symmetric and has the chain cover property. Then build the construction as in Rules 1 and 2, and repeatedly apply Observation 3.1 and Lemmas 3.2 and 3.4, with the induction over n. □

In general there are many RSCDs for a given n, and others can be easily found by hand; this construction, starting from the base case shown for $n = 2$, gives us just one of them.

3.1 Counting Results

In this section we count the number of chains in the RSCD construction given in the previous section. It is easy to prove that the de Bruijn decomposition has the number of chains equal to the middle binomial coefficient $\binom{n}{\lfloor n/2 \rfloor}$ (see [13]).

Let $N(n,k)$ be the number of chains in the construction of length k. Also define the total number of chains in the construction to be $N(n) = \Sigma_{1 \leq k \leq n} N(n,k)$. Directly from the recursive construction, we have the recurrence relation

$$N(n,k) = \begin{cases} 2 & \text{if } n = k \geq 1 \\ N(n-1,2) & \text{if } k = 1 \\ N(n-1,k-1) + N(n-1,k+1) & \text{if } k > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.6. *If n and k have the same parity, then*

$$N(n,k) = 2\frac{k}{n}\binom{n}{(n-k)/2}$$

and, otherwise, $N(n,k) = 0$.

Proof. This recursive formula is the same as that for the Catalan triangle [18, Sequence A053121], with double the base case. The reference [18, Sequence A053121] gives the closed form

$$N(n,k) = \frac{k}{n}\binom{n}{(n-k)/2}$$

for the Catalan triangle, and twice this is the formula we want. □

Corollary 3.7. $N(n,1) = 2\mathbf{C}_{(n-1)/2}$*, twice the $(n-1)/2$th Catalan number, if n is odd, otherwise it is 0.*

Proof. If n is odd, let $m = 2n+1$. Then

$$\begin{aligned} N(n,1) &= \frac{2}{n}\binom{n}{(n-1)/2} \qquad \text{by the previous lemma,} \\ &= 2\frac{1}{2m+1}\binom{2m+1}{m} \\ &= 2\frac{1}{m+1}\binom{2m}{m}, \end{aligned}$$

which is twice a well-known expression for the Catalan number $\mathbf{C}_m$. If n is even, the parity of n and k differ, so by the previous lemma $N(n,1) = 0$. □

Lemma 3.8. *If n is even, all chains in the chain decomposition given by the RSCD construction for n have even length, otherwise all chains have odd length.*

Proof. By induction. Figure 2 shows it is true for $n = 2$, and it can be easily observed to be true for $n = 3$ by applying the construction. Assume the lemma is true for $n-1$ and $n-2$. If n is even, assume the construction for $n-1$ has all odd-length chains. The replacement rule replaces a chain of length k, with k odd, with two chains of length $k+1$ and $k-1$ respectively, so their lengths must both be even, or one chain of length two if $k = 1$. In both cases the resulting chains all have even length. The case for n odd is similar. □

Theorem 3.9. *The number of chains $N(n)$ in the RSCD construction for $n > 1$ is $\binom{n}{n/2}$ if n is even, $2\binom{n-1}{(n-1)/2}$ otherwise.*

Proof. By induction. The basis for $n = 2$ is shown in Figure 2 and the construction for $n = 3$ is easily generated. We consider the two cases, n odd and even.

For n odd, the number of chains in the construction for $n-1$ is $\binom{n-1}{(n-1)/2}$ as $n-1$ is even. By Lemma 3.8 all chains in the construction for $n-1$ are of even length and so are all replaced by two odd chains; from the construction it is apparent that each even-length chain gives exactly two odd-length chains. Thus for n odd, $N(n) = 2\binom{n-1}{(n-1)/2}$.

For n even, the construction for $n-1$ gives $2\binom{n-2}{(n-2)/2}$ chains by induction, and all chains are odd length. Applying the construction rule gives two chains for every chain in the construction for $n-1$, except for chains of length one which will give one chain. By Lemma 3.7 the number of length one chains is

$$N(n-1,1) \;=\; 2\mathbf{C}_{\frac{n-2}{2}} \;=\; \frac{2}{(n-2)/2+1}\binom{n-2}{(n-2)/2} \;=\; \frac{4}{n}\binom{n-2}{(n-2)/2} \;.$$

Thus the number of chains for n even is

$$\begin{aligned} N(n) &= 2N(n-1) - N(n-1,1) \\ &= 2\cdot 2\binom{n-2}{(n-2)/2} - \frac{4}{n}\binom{n-2}{(n-2)/2} \\ &= 4(\frac{n-1}{n})\binom{n-2}{(n-2)/2}, \text{ which is easily reduced to} \\ &= \binom{n}{n/2}, \text{ as desired.} \end{aligned}$$

□

It is well-known that the minimum number chains possible in a chain decomposition is $\binom{n}{\lfloor n/2\rfloor}$. The previous results have shown that the construction may, depending on n, give a chain decomposition with more than the minimum number of chains possible.

Corollary 3.10.

$$N(n) - \binom{n}{\lfloor n/2\rfloor} \;=\; \mathbf{C}_{\frac{n-1}{2}} \text{ if } n \text{ odd, } n > 2 \;,$$

that is, when n is odd, the difference between the number of chains produced by the PSCD construction and the minimum number of chains in a chain decomposition of $\mathcal{B}_n$ is the $(n-1)/2$th Catalan number.

Proof. If n is odd,

$$N(n) - \binom{n}{\lfloor n/2\rfloor} \;=\; 2\binom{n-1}{(n-1)/2} - \binom{n}{(n-1)/2}$$

which is easily reduced to $\mathbf{C}_{\frac{n-1}{2}}$. □

4 Antipodally Symmetric Chain Decomposition Embeddings

In this section we show how the construction for decompositions given in the previous section can be easily modified to give a chain decomposition that has symmetry under an antipodal map, which we refer to as an *antipodally symmetric chain decomposition embedding* (ASCD).

Let E be a chain decomposition embedding of $\mathcal{B}_n$ with k chains, k an even number, with chains numbered sequentially (as described in Section 2.1). Then E is *antipodally symmetric* if the function $f(C_i) = C_{i+k/2 \mod k}$ swapping the first and second halves of the chains, and applying $\phi(x) = \overline{x}$ to the elements is a dual-automorphism of the embedding. We define the following construction recursively.

Basis. The basis for a recursive construction is an ASCD for $\mathcal{B}_2$, shown in Figure 2 in the previous section.

The modification of the earlier construction simply consists of reordering the rows in the second half of the embedding; the chains in the ASCD construction are the same as in the RSCD construction.

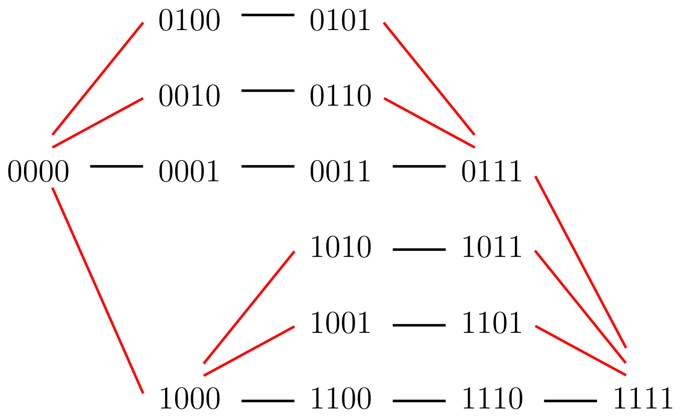


Figure 4: Chain decomposition embedding symmetric under antipodal map, generated by the recursive construction, for $\mathcal{B}_4$; chain cover edges are shown in red

Similar to the rules in the previous section, the following recursive rules allow us to construct, given an ASCD for the n-cube $\mathcal{B}_n$, an ASCD for the $(n+1)$-cube $\mathcal{B}_{(n+1)}$.

Recursive Rules. Given an ASCD for $\mathcal{B}_n$ with m rows, the ith row '$\sigma_1\ \sigma_2\ \ldots\ \sigma_k$' is replaced in one of two ways, depending on whether the row is in the first half of the ASCD (*i.e.* whether $i \leq \frac{m}{2}$).

Rule 3. If $i \leq \frac{m}{2}$, the row becomes the two rows

$$\begin{array}{ccccc} & \sigma_2 0 & \ldots & \sigma_k 0 & \\ \sigma_1 0 & \sigma_1 1 & \ldots & \sigma_{k-1} 1 & \sigma_k 1, \end{array}$$

where if $k = 1$ the first row is omitted and the second becomes $\sigma_1 0\ \sigma_1 1$; this is identical to the first rule used in the recursive construction for RSCDs in the previous section.

Rule 4. If $i > \frac{m}{2}$, the row becomes

$$\begin{array}{ccccc} & \sigma_1 1 & \ldots & \sigma_{k-1} 1, & \\ \sigma_1 0 & \sigma_2 0 & \ldots & \sigma_k 0 & \sigma_k 1 \end{array}$$

where if $k = 1$ the first row is omitted and the second becomes $\sigma_k 0\ \sigma_k 1$. This is similar to the second rule for RSCDs in the previous section, except the order of the rows is reversed.

It is easy to prove, in the same way as the previous section, that the decomposition has all the right properties.

Observation 4.1. *Given an ASCD M for $\mathcal{B}_n$ and applying the construction to give an ASCD M' for $\mathcal{B}_{(n+1)}$, chains embedded as rows in M become chains embedded as rows in M'.*

Proof. By definition of Rules 3 and 4. $\square$

Lemma 4.2. *Given an ASCD M for $\mathcal{B}_n$ and applying Rules 3 and 4 to give an ASCD M' for $\mathcal{B}_{(n+1)}$, if M has antipodal symmetry, M' does also.*

Proof. The proof is almost identical to that of Lemma 3.2, except that the order of the rows in Equation 3 switch. $\square$

Lemma 4.3. *Given an ASCD M for $\mathcal{B}_n$ and applying Rules 3 and 4 to give an ASCD M' for $\mathcal{B}_{(n+1)}$, if M has the chain cover property, then M' also has the chain cover property.*

Proof. The proof is almost identical to that of Lemma 3.4 above. The chain cover edges shown in the proof of Lemma 3.4 change to reorder the chains in the application of Rule 4.

Given the chain

$$\begin{tikzcd}
a \arrow[dr, dash] & & & & & \\
& \sigma_1 \arrow[r, dash] & \sigma_2 \arrow[r, dash] & \ldots \arrow[r, dash] & \sigma_k \arrow[dr, dash] & \\
& & & & & b
\end{tikzcd}$$

Rule 3 gives the chains:

$$\begin{tikzcd}
a0 \arrow[dr, dash] & & \sigma_2 0 \arrow[r, dash] & \ldots \arrow[r, dash] & \sigma_k 0 \arrow[dr, dash] & & \\
& \sigma_1 0 \arrow[ur, dash] \arrow[r, dash] & \sigma_1 1 \arrow[r, dash] & \ldots \arrow[r, dash] & \sigma_{k-1} 1 \arrow[r, dash] & \sigma_k 1 \arrow[r, dash] & b1
\end{tikzcd}$$

and Rule 4 gives the chains:

$$\begin{tikzcd}
a0 \arrow[dr, dash] & & \sigma_1 1 \arrow[r, dash] & \ldots \arrow[r, dash] & \sigma_{k-1} 1 \arrow[dr, dash] & & \\
& \sigma_1 0 \arrow[ur, dash] \arrow[r, dash] & \sigma_2 0 \arrow[r, dash] & \ldots \arrow[r, dash] & \sigma_k 0 \arrow[r, dash] & \sigma_k 1 \arrow[r, dash] & b1
\end{tikzcd}$$

□

Theorem 4.4. *There exists a chain decomposition embedding of $\mathcal{B}_n$ that is symmetric under an antipodal map, for all $n \geq 2$, such that*

- *the number of chains is $N(n)$ (see Theorem 3.9),*
- *the chain decomposition embedding has the chain cover property.*

Proof. The basis in Figure 2 is antipodally symmetric and has the chain cover property. Then build the construction by inductively applying Rules 3 and 4, and repeatedly apply Observation 4.1 and Lemmas 4.2 and 4.3, with induction over n. □

All of the counting results from Section 3.1 still apply, since the only difference in the construction is to reorder half of the chains in the embedding.

5 Diagrams with Order Two Isometries from Chain Decomposition Embeddings

In this section we discuss the Venn diagrams that are created on the sphere by taking the dual diagrams of the chain decomposition embeddings discussed in Sections 3 and 4. These constructions give diagrams with total symmetries (*i.e* all curves map onto themselves) realizing the first two non-trivial involutions on the sphere.

The following lemma is part of the paper [9]:

Lemma 2([9, p. 8])**.** *Let $\mathcal{C}$ be a symmetric chain decomposition with the chain cover property for $\mathcal{B}_n$, let π be a chain cover mapping for $\mathcal{C}$, and let $P(\mathcal{C}, \pi)$ be the planar embedding of $G(\mathcal{C}, \pi)$ [described in the proof of a related lemma]. Then the geometric dual of $P(\mathcal{C}, \pi)$, called $P^*(\mathcal{C}, \pi)$, is a monotone Venn diagram of n curves with the minimum number of vertices.*

This result, henceforth referred to as the "GKS construction", can easily be changed to accommodate chain decompositions embedded on the surface of the sphere, as follows in subsequent sections. First, we enhance the definition of the embeddings: a *spherical chain decomposition embedding* of a chain decomposition $\mathcal{C}$ with the chain cover property is an embedding $P(\mathcal{C}, \pi)$ of the chain cover graph $G(\mathcal{C}, \pi)$ on the 2-sphere.

It is clear that any graph G with a planar embedding also has an embedding on the sphere, provided, if a cylindrical projection is used, the planar embedding of G does not map more than one point onto each line which is an image of a pole on the sphere. We will use reverse cylindrical projections in the following subsections to produce particular spherical chain decomposition embeddings of the chain decomposition embeddings described in the previous sections, and then show that the diagrams produced have some symmetries not present on the plane.

5.1 Venn Diagrams with Rotational Symmetry on the Sphere

In this section we discuss the diagrams created as the duals of the reverse symmetric chain decomposition embeddings discussed in Section 3.

The following proof establishes the resulting symmetry from the n-Venn diagrams on the sphere created from the construction of the previous section.

Lemma 5.1. *Given n, let $\mathcal{C}$ be a RSCD for $\mathcal{B}_n$ with the chain cover mapping π and the planar embedding $P(\mathcal{C}, \pi)$. Then there exists a vector $\vec{v}$ such that $P(\mathcal{C}, \pi)$ has an embedding on the 2-sphere such that the geometric dual of $P(\mathcal{C}, \pi)$, called $P^*(\mathcal{C}, \pi)$, is a monotone Venn diagram of n curves with total symmetry under the rotation $R_{\vec{v},\pi}$.*

Proof. We first describe the embedding of $\mathcal{C}$ onto the plane, the emdedding of its labelled, dual graph, and then specify the projection onto the sphere giving the final diagram. On the plane, the ith chain $C_i \in \mathcal{C}$ is embedded vertically (with all vertices on a line $x = i$) such that a vertex x on C has y-coordinate $-rank(x)$ and x-coordinate i. All edges of $\mathcal{C}$ are embedded as straight lines. The chains and chain cover edges are thus ordered horizontally as in the construction of Section 3, from left to right in increasing order of i, with all nodes of equal rank at the same horizontal level, and edges embedded vertically; the chain cover edges are embedded as straight lines between nodes differing in rank by one. Note that this embedding is essentially the same as that of the GKS construction [9] except with a lack of mirror symmetry between the chain cover edges. The line $y = -rank(C)/2$, *i.e* the horizontal line through the elements of rank $n/2$ (if n is even) or halfway between the elements of rank $(n-1)/2$ and $(n+1)/2$ (if n is odd), is referred to as the *equator* of the embedding.

We next construct the geometric dual $P^*(\mathcal{C}, \pi))$ of $\mathcal{C}$, noting that we overload the notation P^* to refer to the planar dual of $P(\mathcal{C}, \pi)$ as well as the dual of the embedding on the sphere of $P(\mathcal{C}, \pi)$; it will be clear from context which surface embedding is being referred to. The dual on the plane is also constructed as in the GKS construction. With one exception, discussed in the following paragraph, vertex f^* of P^* is placed in the interior of face f of P, such that the x-coordinate of f is $i + 1/2$, where C_i and C_{i+1} are the two chains embedded on either side of f, and the y-coordinate of f^* is such that it lies on the equator. We can refer to the vertices $f^* \in P^*$ that are embedded in the faces of P on the equator as *face vertices*.

Let $m = N(n)$ be the number of chains. Since the chains are spaced equally according to their index by the construction, the leftmost, or first, chain, with minimum x-coordinate, is C_1 and the rightmost, or last, is C_m. Let t refer to the external face on the plane; embed the vertex $t^* \in P^*$ at the point on the equator with x-coordinate $1/2$, and also embed an image of t^* on the equator at x-coordinate $m + 1/2$. These two images will be identified under the reverse cylindrical projection onto the sphere that will be performed on P^* applied to the rectangle bounded to either side by the lines $x = 1/2$ and $x = m + 1/2$.

For example, Figure 5(a) shows the construction of P^*, with edges drawn as smooth curves for clarity. The images of t^* are identified as v' in Figure 5.

To construct the edges of $P^*(\mathcal{C}, \pi)$, a point is chosen on each edge of P where the edge of P^* will cross it. For each face f of P and each edge e on on its boundary, a half-edge of P^* is drawn from f^* to the crossing point so that the half-edges incident with f^* are internally disjoint. The two half-edges of P^* meet at the crossing point and together form the edge

$e^* \in P^*$, and e^* is labelled with the index x of the bit that changes between the endpoints of e, so $e = (S, S \cup \{x\})$. Edges incident to the extremal point t^* are drawn such that edges crossing chain C_1 and its cover edges are incident to the image of t^* at $x = 1/2$, and edges crossing C_m and its cover edges are incident to the image of t^* at $x = m + 1/2$;

Now construct the reverse cylindrical projection onto the sphere of $P^*(\mathcal{C}, \pi)$ about a rectangle bounded by $x = 1/2$, $x = m + 1/2$, $y = 0$, and $y = -n$; define the projection by $\theta = \frac{2\pi(x-1/2)}{m}$ and $\phi = \pi y/n + \pi/2$. This projection maps the two images of t^* onto the point at $(\theta, \phi) = (0, 0)$ and the equator maps to the sphere's equatorial line $\phi = 0$. The face that is the dual of the vertex of P corresponding to the empty set, labelled 0^n, includes the north pole at $\phi = \pi/2$, and the face corresponding to the complete set, labelled 1^n, contains the south pole at $\phi = -\pi/2$.

Using the proof of the GKS construction [9] it is straightforward to show that $P^*(\mathcal{C}, \pi)$ on the sphere has the properties required of an n-Venn diagram; to show that the edges labelled i in P^* form a simple cycle and thus form a simple closed curve the technique is to show that the corresponding edges of P on the sphere form a *bond*, that is, a minimal set of edges whose removal disconnects P. Furthermore P^* is an n-Venn diagram on the sphere, which can be shown by considering the 2^n vertices of P that correspond to the 2^n faces of P^*. Also note that in P each vertex of $rank(k)$, $0 < k < n$, is adjacent to a vertex of rank $k-1$ and a vertex of rank $k+1$ which implies that in P^* each region of weight k, where $0 < k < n$, is adjacent to a region of weight $k-1$ and a region of weight $k+1$, and thus $P^*(\mathcal{C}, \pi)$ is a monotone n-Venn diagram.

We now consider the symmetries of P^* on the sphere. Define the vector $\vec{v}$ from the centre of the sphere through the point $v = (\theta = \pi, \phi = 0)$ on the equator and antipodal to the image v' of t^* at $(0, 0)$; this vector defines an axis of rotation through the points v, v'. This axis passes through the face vertex $f^* \in P^*$ that is embedded in the face between the two chains $C_{m/2}$ and $C_{m/2+1}$, and as noted, through the vertex $v' \in P^*$ between the chains C_1 and C_m. The rotational symmetry of period two about this axis is $R_{\vec{v},\pi}$. The action of $R_{\vec{v},\pi}$ on a point $p = (\theta, \phi)$ is $R_{\vec{v},\pi}(p) = (2\pi - \theta, -\phi)$.

Now consider a non-terminator vertex $p \in C_i$ with label $p = p_1 p_2 \ldots p_n$ corresponding to a set S; it is embedded on the planar embedding P at position $(x, y) = (i, -rank(p))$, and the embedding on the 2-sphere puts p at position

$(\theta, \phi) = (\frac{2\pi(i-1/2)}{m}, \frac{-\pi rank(p)}{n} + \pi/2)$.

Consider a neighbouring vertex to p, called q, on the same chain C_i, with $rank(q) = rank(p) + 1$ and labelled corresponding to set $S \cup \{x\}$; on the sphere q is at position

$(\theta, \phi) = (\frac{2\pi(i-1/2)}{m)}, \frac{-\pi(rank(p)+1)}{n} + \pi/2)$.

The edge $e = (p, q)$ is embedded as an arc on the line of longitude $\theta = \frac{2\pi(i-1/2)}{m}$, and the edge e^* of P^*, with $label(e^*) = x$, crosses e at the chosen point (the image under the projection of the crossing point chosen to build P^* on the plane).

Under the operation of $R_{\vec{v},\pi}$, the points p and q are taken to

$$\begin{aligned} R_{\vec{v},\pi}(p) &= (2\pi - \frac{2\pi(i-1/2)}{m}, -(\frac{-\pi\, rank(p)}{n} + \pi/2)) \\ &= (\frac{2\pi(m-i+1/2)}{m}, \frac{\pi\, rank(p)}{n} - \pi/2) \\ \text{and} & \\ R_{\vec{v},\pi}(q) &= (\frac{2\pi(m-i+1/2)}{m}, \frac{\pi(rank(p)+1}{n} - \pi/2)\ . \end{aligned}$$

Now consider the reverse symmetry of the embedding P, in which the operation of reversing the chains, reversing the order of the chains, and complementing the elements of the chain is an automorphism of the embedding. Thus, the vertex $p \in C_i$ under the reverse symmetry on the chain decomposition embedding maps to the point p' on the chain C_{m-i+1} with $rank(p') = n - rank(p)$, and the set corresponding to the label of p' is $\overline{S}$. The position of p' in P^* on the plane is $(x, y) = (m - i + 1, -(n - rank(p))) = (m - i + 1, rank(p) - n)$. Now, under the cylindrical projection onto the 2-sphere, p' is at position

$$\begin{aligned} (\theta, \phi) &= (\frac{2\pi(m-i+1-1/2)}{m)}, \frac{\pi(rank(p)-n)}{n} + \pi/2) \\ &= (\frac{2\pi(m-i+1/2)}{m)}, -\pi + \frac{\pi(rank(p))}{n} + \pi/2) \\ &= (\frac{2\pi(m-i+1/2)}{m)}, \frac{\pi(rank(p))}{n} - \pi/2)\ . \end{aligned}$$

Similarly, the image of q, called q', projects to

$$(\theta, \phi) = (\frac{2\pi(m-i+1/2)}{m)}, \frac{\pi(rank(p)+1)}{n} - \pi/2)\ ,$$

and so $R_{\vec{v},\pi}(p) = p' \in P^*$ and $R_{\vec{v},\pi}(q) = q' \in P^*$.

A similar argument holds if (p, q) is a chain cover edge, the only difference in the above is that the y-coordinate for $q \in P$ is not the same as that of p, so the ϕ-coordinate of $q \in P$ on the sphere is different than for p.

The points chosen for the crossing points of the edges in P^* have the same symmetry as they are chosen consistently in the chain edges and chain cover edges throughout P. Recall also the face vertices f^* chosen on the equator as vertices in P^*, where f^* has x-coordinate $\frac{i+(i+1)}{2}$, where C_i and C_{i+1} are the chains on either side of f^*. These points are the endpoints of the edges $e^* \in P^*$, and since on the sphere the point f^* is located at coordinates $(\theta, \phi) = (\frac{2\pi(2i+1/2)}{2m}, 0)$, then $R_{\vec{v},\pi}(f^*) = g^*$, where g^* is some similar face vertex ($g^* = f^*$ itself if $i = m/2$, otherwise it is a different face vertex).

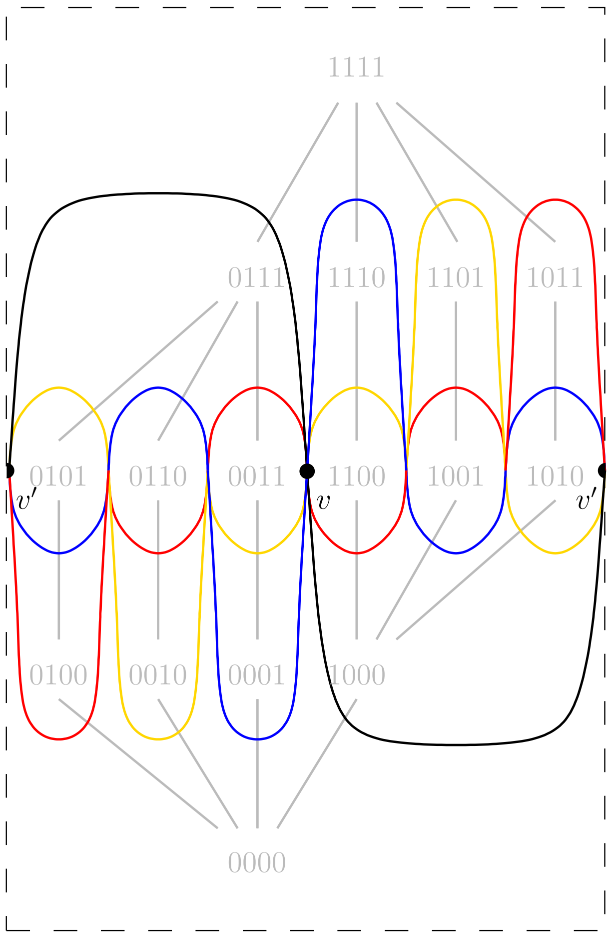
Now consider the edge $e^* \in P^*$ with $label(e^*) = x$ such that p has the label corresponding to subset S and q has the label corresponding to subset $S \cup \{x\}$.

The edge $e^{*'} \in P^*$ on the sphere that is the image of e^* under $R_{\vec{v},\pi}$ has some label, call it y. Let $label(R(p)) = T$ and $label(R(q)) = T - \{y\}$, for some subset T. By the definition of the chain decomposition embedding, $label(R(p)) = \overline{S}$ and $label(R(q)) = \overline{S \cup \{x\}}$. Let

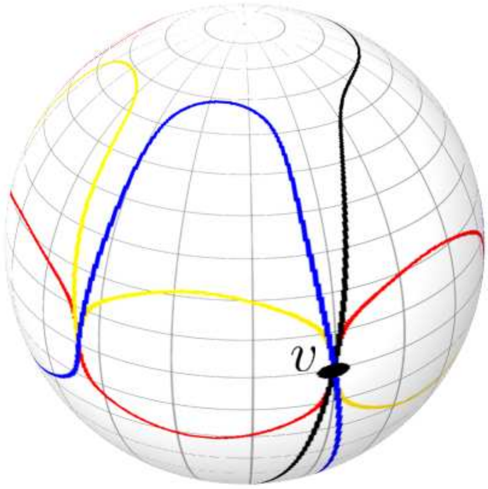
$T = \overline{S}$; then $label(R(q)) = \overline{S \cup \{x\}} = T - \{x\}$, and so $y = x$. In other words, the bit that changes over the edge $(R(p), R(q)) = e^{*'} \in P^*$ is x, and so $label(e^{*'}) = label(e^*) = x$.

We have established that the image of endpoints of e^*, which are the face vertices, are face vertices also, and the point where e^* crosses the edge (p, q) is preserved in the edge $(R_{\vec{v},\pi}(p), R_{\vec{v},\pi}(q))$, and if the edges are drawn consistently throughout the embedding then the arcs joining these points will be preserved also, and finally we have established that the label of the image $e^{*'}$ of e^*, which is the colour of the edge, is the same as e^*. This establishes that the image of an edge of P^* under $R_{\vec{v},\pi}$ is another edge of the same colour (for example, see Figure 5(a)). This establishes that $R_{\vec{v},\pi}$ is a total symmetry for P^*. $\square$

Figure 5 shows the diagram constructed from the RSCD construction from Section 3, with the curves drawn as smooth curves for aesthetics. The axis of rotation for the rotational symmetry passes through the vertices v and v' on opposite sides of the sphere at the equator.



(a) Cylindrical projection of diagram



(b) Diagram on sphere

Figure 5: 4-Venn diagram with rotational symmetry from the RSCD for $\mathcal{B}_4$, with axis of rotation passing through vertices v and v'

5.2 Venn Diagrams with Inversion Symmetry on the Sphere

In this section we discuss the diagrams created as the dual of the antipodally symmetric chain decomposition embeddings discussed in Section 4.

The following lemma establishes that the n-Venn diagrams on the sphere created from the construction of the previous section have antipodal symmetry. An interesting situation that does not occur with the reverse symmetric diagrams (see the proof of Lemma 5.1) is that if the two adjacent vertices p and q in the antipodally symmetric chain decomposition embedding are such that (p, q) is a chain cover edge and $label(p) = 0^n$, then q may be such that the edge (p, q), under the glide reflection $G_{\vec{v},\pi}((p, q))$ on the sphere (as we discuss in the proof), crosses the line of longitude at $\theta = 0$. Representing this edge on the embedding of P^* in the plane is awkward, and we have always chosen to break the symmetry in the planar representation to preserve continuity of this chain cover edge by drawing it in a continuous fashion. This implies that if the cylindrical projection is taken of P^* on the sphere as we have described it *back* onto the plane, the edge $(G_{\vec{v},\pi}(p), G_{\vec{v},\pi}(q))$ would cross the edge of the enclosing rectangle, and so we have always chosen to draw it to be contained in that rectangle. For example, in Figure 6(a) the edge $(0111, 1111)$ is drawn contiguously within the inscribing rectangle instead of angling downwards to the left from 0111, across the rectangle boundary, to 1111. Note again that this breaking of symmetry does not apply to the spherical embedding of P^* and so it does not affect the following proof.

Lemma 5.2. *Given n, let $\mathcal{C}$ be an ASCD for $\mathcal{B}_n$ with the chain cover mapping π and the planar embedding $P(\mathcal{C}, \pi)$. Then $P(\mathcal{C}, \pi)$ has an embedding on the 2-sphere such that the geometric dual of $P(\mathcal{C}, \pi)$, called $P^*(\mathcal{C}, \pi)$, is a monotone Venn diagram of n curves with total symmetry under the glide reflection $G_{\vec{v},\pi}$ (also called inversion symmetry), for any vector $\vec{v}$.*

Proof. This proof is almost identical to the proof of Lemma 5.1, since the constructions and resulting diagrams are very similar.

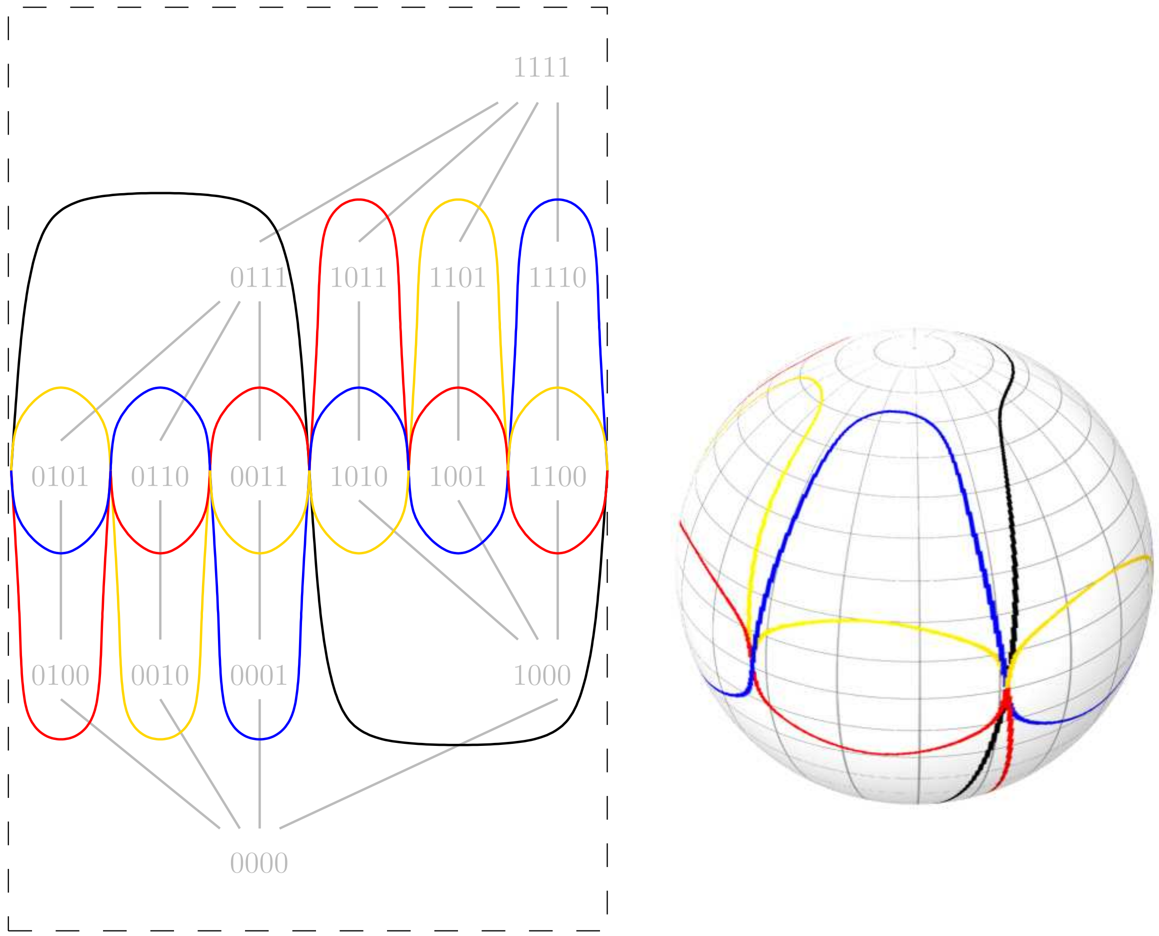
The construction on the plane and reverse cylindrical projection are the same, taking into account the differences in the chain layout for the antipodal symmetric chain decomposition embedding, and the resulting diagram is a monotone n-Venn diagram on the sphere.

The key difference for the remainder of the proof is that to define the symmetry $G_{\vec{v},\pi}$, any vector v can be used, but for convenience we can choose the polar axis as is our usual convention, such that $\vec{v}$ is defined from the centre of the sphere through the point $\phi = \pi/2$ at the north pole. Thus the action of $G_{\vec{v},\pi}$ on a point $p = (\theta, \phi)$ is $G_{\vec{v},\pi}(p) = (\theta + \pi, -\phi)$. Defining the vertices p and q as in the proof of Lemma 5.1, it is easy to verify that the image of p under the antipodal symmetry of the chain decomposition embedding, and the image p' of p under $G_{\vec{v},\pi}$, is the point

$$G_{\vec{v},\pi}(p) = (\frac{2\pi(i - 1/2)}{m} + \pi, \frac{\pi rank(p)}{n} - \pi/2)) \ ,$$

with a similar equation for q.

The same considerations of the images of the points in P^* hold for the edge $e^* \in P^*$ between p and q, and the same arguments show that the endpoints of $e^{*\prime}$ are preserved, the crossing point is preserved, the arcs are preserved, and $label(e^*) = label(e^{*\prime})$, and thus the inversion operation $G_{\vec{v},\pi}$ is a total symmetry for P^*. $\square$



(a) Cylindrical projection of diagram

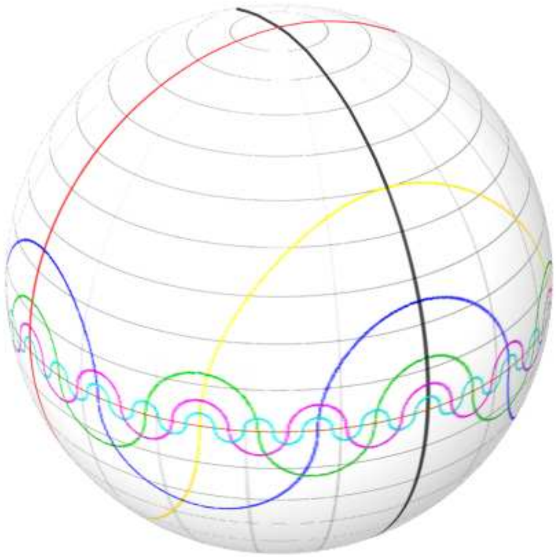
(b) Diagram on sphere

Figure 6: Antipodally symmetric 4-Venn diagram from the ASCD for $\mathcal{B}_4$

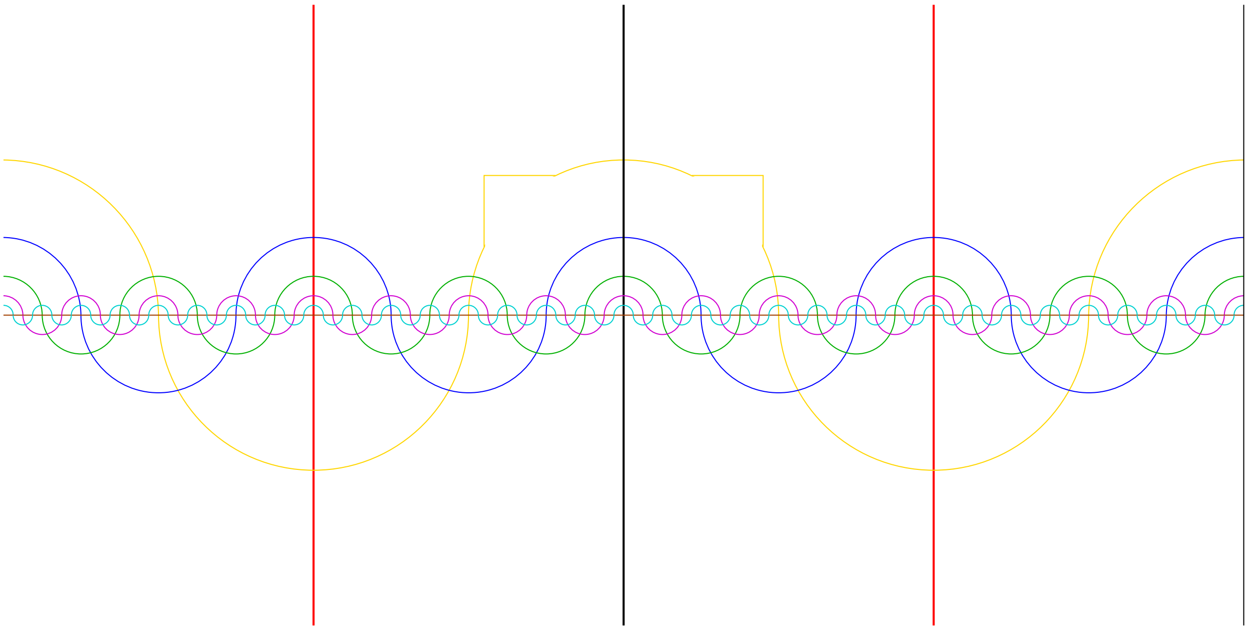
Figure 6 shows the diagram constructed from the ASCD construction from Section 4.

6 Reflective Symmetric Venn Diagrams on the Sphere

The third symmetry of order two that we wish to realize on the sphere is reflection. For this we will use the construction by Anthony Edwards, in a series of papers [5, 7], for any n, of an n-Venn diagram on the plane. Edwards' construction is isomorphic (up to curve relabelling and continuous deformation of the plane) to earlier constructions such as Grünbaum's construction from 1975 [10], and an even earlier construction by Anusiak [1]. Edwards was the first to note the possibility of observing interesting symmetries of the diagrams produced by embedding them on the sphere, and in this section we expand and formalize this idea.



(a) Diagram on the sphere



(b) Cylindrical projection, using semicircular arcs to construct curves for $n \geq 4$

Figure 7: The Edwards construction for an n-Venn diagram, drawn on the sphere

Figure 7 shows the Edwards construction up to $n = 8$ curves on the sphere, and the cylindrical projection, drawn using circular arcs. The first three curves in the construction are the equatorial curve and the two orthogonal great circles passing through the poles, with the third curve considered to be the red curve. Each subsequent curve is a smaller and repeated copy of the previous, bisecting each region at a point on or close to the equator.

It is clear from the figures that on the sphere the diagram has reflective total symmetries across the orthogonal vertical planes containing the two curves passing through the poles; thus for any n this diagram has two reflection isometries.

The Edwards construction also has an interesting variant, called *binary-form Edwards diagrams*, which proves to have isometries related to our earlier constructions. Their name arises from their connections to Gray codes, which can be found in the diagram by considering the labelled dual graph—see Edwards' publications [4] and [5, Chapter 5] for more details. To create the diagrams, from the diagram in Figure 7, the ith curve is rotated about the polar axis by $\pi/2^{i-2}$ radians (in the cylindrical projection they are shifted horizontally, where the equatorial line has length 2π). The third curve, coloured red in Figure 7, becomes identified with the second (black) curve, but the other curves remain distinct. The resulting diagram is shown in Figure 8.

On the sphere the binary-form diagram has three rotational total symmetries $R_{\vec{v},\pi}$ about orthogonal axes: the vertical (polar axis) coinciding with the z axis, and the x and y axes shown in Figure 8(b) passing through (x, x') and (y, y') respectively. Thus these diagrams also realize a rotation isometry, like our construction for reverse symmetric diagrams.

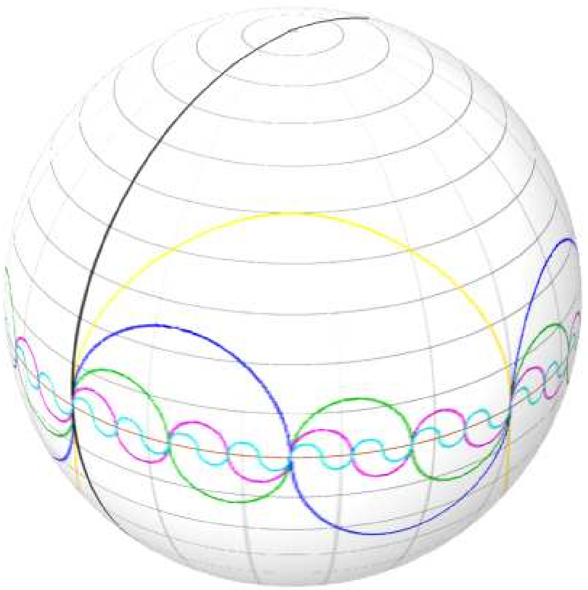
7 Further Work

We noted in Section 3.1 that the construction given for reverse symmetric and antipodally symmetric chain decompositions gives more than the minimum number of chains, starting with $n \geq 5$. The chain decomposition embedding with antipodal symmetry shown in Figure 9 for $n = 5$ contains $\binom{5}{\lfloor 5/2 \rfloor} = 10$ chains, as opposed to the number $N(5) = 12$, from Subsection 3.1, given by the construction. Of course, a reverse symmetric chain decomposition can be easily created from that in Figure 9 by reversing the order of the second half of the chains. Since this construction has the chain cover property, it then gives another 5-Venn diagram with the symmetries described in Lemmas 5.1 and 5.2.

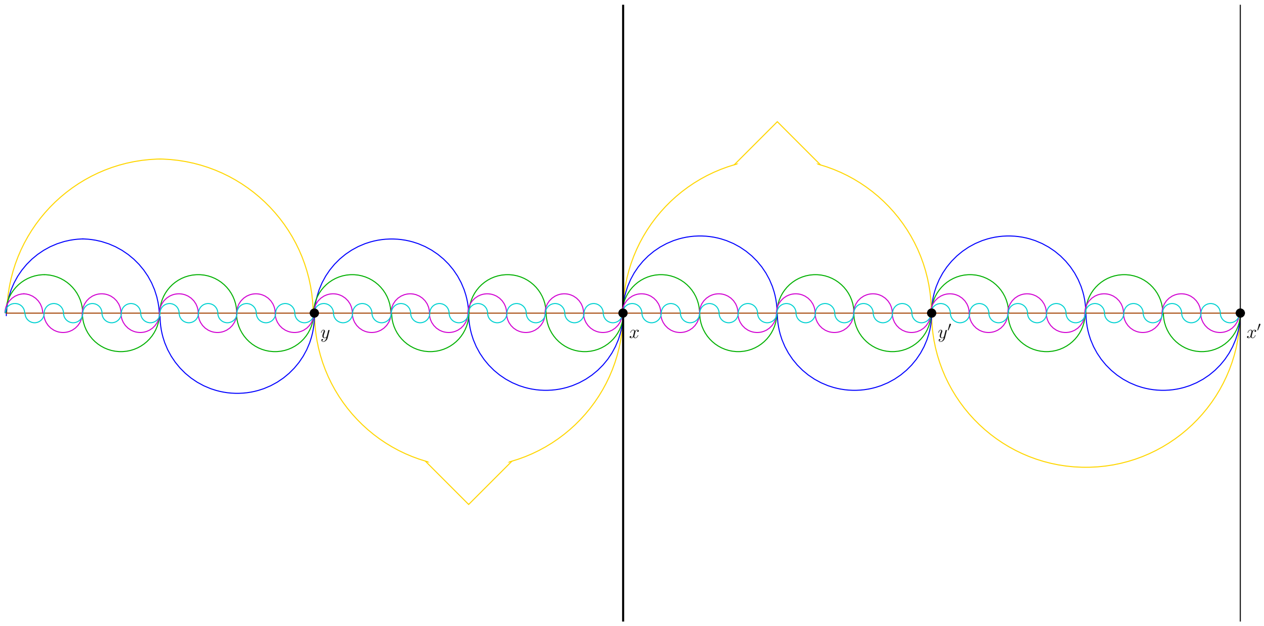
A natural question to ask is whether there exist reverse symmetric (or antipodally symmetric) chain decompositions for any n odd with $\binom{n}{(n-1)/2}$ chains. If so, a construction similar to that for RSCDs (or ASCDs) would be interesting.

Furthermore, it is easy to see that the resulting diagrams from our construction, as developed in Section 5, are not simple, as usually the centre point between chains $C_{m/2}$ and $C_{m/2+1}$ (the point of rotation in the reverse symmetric diagrams) is a vertex of degree $2n$ in the resulting n-Venn diagram. Simple diagrams of this type seem to be hard to find, and it would be interesting to investigate when, or if, they exist.

The results of this paper have established that, for any $n > 1$, any of the order-two isometries on the sphere have a corresponding Venn diagram of order n that have that isometry. In future papers [16, 17] the authors will present further work which will discuss all possible symmetry groups on the sphere, and discuss many more Venn diagrams realizing these other groups.



(a) Diagram on the sphere



(b) Cylindrical projection

Figure 8: Edwards construction modified to binary-form, drawn on the sphere

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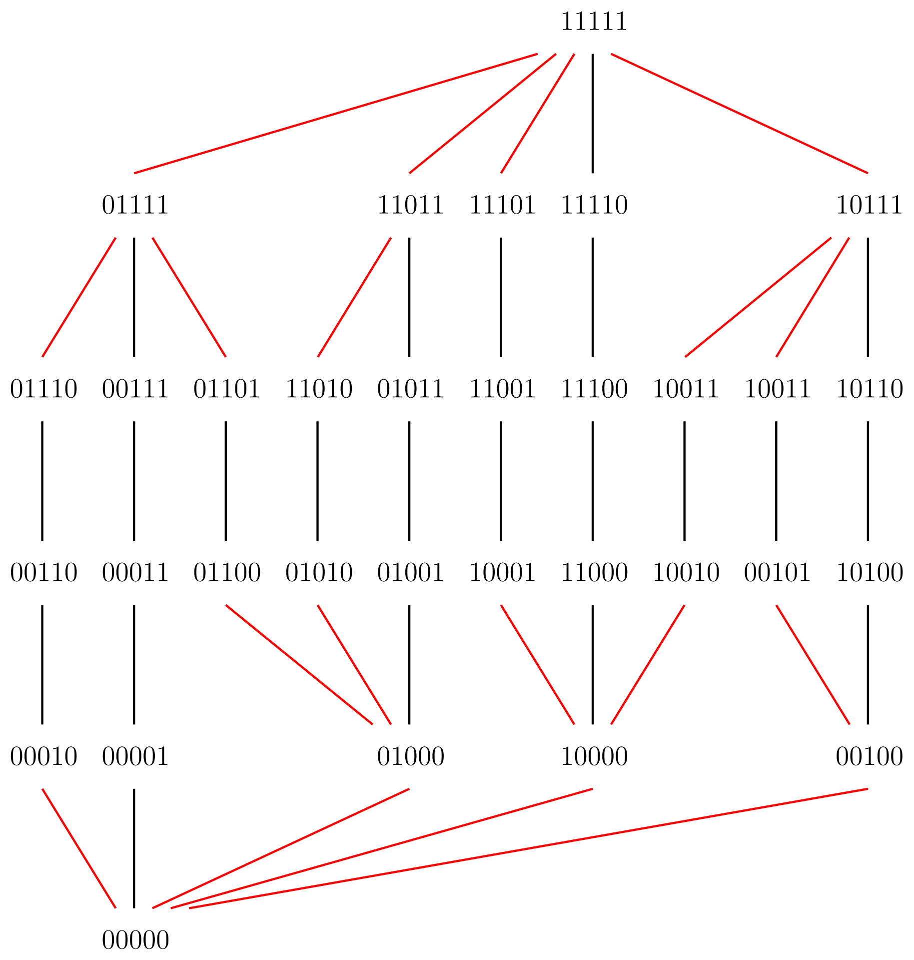

Figure 9: ASCD of $\mathcal{B}_5$ with minimum number of chains

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