# COUNTING STRINGS WITH GIVEN ELEMENTARY SYMMETRIC FUNCTION EVALUATIONS I: STRINGS OVER $\mathbb{Z}_P$ WITH $P$ PRIME

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**Abstract.** Let $\alpha$ be a string over $\mathbb{Z}_p$ with $p$ prime. The $j$-th elementary symmetric function evaluated at $\alpha$ is denoted $T_j(\alpha)$. We study the cardinalities $S_p(n; \tau_1, \tau_2, \ldots, \tau_t)$ of the set of length $n$ strings for which $T_i(\alpha) = \tau_i$. The *profile* $\langle k_0, k_1, \ldots, k_{p-1}\rangle$ of a string $\alpha$ is the sequence of frequencies with which each letter occurs. The profile of $\alpha$ determines $T_j(\alpha)$, and hence $S_p$. Let $f_n : \mathbb{Z}_{p^n}^{p-1} \mapsto \mathbb{Z}_p^{p^n-1}$ be the map that takes $\langle k_0, k_1, \ldots, k_{p-1}\rangle$ mod $p^n$ to $(T_1, T_2, \ldots, T_{p^n-1})$ mod $p$. We show that $f_n$ is well-defined and injective and how to efficiently determine its range. These results are used to efficiently compute $S_p(n; \tau_1, \tau_2, \ldots, \tau_t)$.

**Key words.** elementary symmetric function, combinatorial enumeration, integers mod a prime.

**AMS subject classifications.** 05A15, 05E05, 05A19

**1. Introduction.** The theory of symmetric functions has long been a basic tool of combinatorial enumeration. Indeed, Cameron [1] states that "one can appreciate the view held by some people, that if it isn't related to symmetric polynomials, then it isn't combinatorics!". However the enumeration of the number of variable assignments to symmetric functions so that the functions achieve given values seems to be new and interesting.

In order for the problem to make sense, we must choose the variables to come from some finite algebraic structure, and pick a particular class of symmetric functions. Here we choose the variables to come from the ring of integers mod a prime $p$, and the class of elementary symmetric functions. The elementary symmetric functions are important since they give the coefficients of polynomials in terms of their roots.

The main purpose of this paper and the companion paper [5] is to count certain strings over the ring of integers mod $p^n$ and over the finite field $\mathbb{F}_{p^n}$, where $p$ is prime. We take the point of view espoused by Wilf [6] that the intrinsic worth of an expression is determined by the amount of computation that it takes to evaluate it. We will state our running times in terms of the number of ring and arithmetic operations that it takes to evaluate the expression using the obvious algorithm. The word size of the computer is assumed to be $O(\log n)$ since the largest numbers we deal with have size $O(r^n)$ where $r$, the cardinality of the ring, is regarded as being a constant.

This work is a continuation of [2] where the number of monic irreducible polynomials over $\mathbb{F}_2$ of degree $n$ with given trace and "subtrace" are enumerated. The trace is the coefficient of $x^{n-1}$ and the subtrace the coefficient of $x^{n-2}$. If such a polynomial is factored in a splitting field, the trace and subtrace can be viewed as the first and second elementary symmetric functions evaluated at the string of coefficients appearing in the factorization. The techniques in [2] are elementary in nature and and involve the relationship of Lyndon words to irreducible polynomials. It therefore seems a natural extension of these ideas to count higher order "traces" on strings with values in various rings.

**2. Preliminaries.** Consider a string $\alpha = a_1 a_2 \cdots a_n$ where each $a_i \in \mathbb{Z}_p$. Define the *j-trace* of $\alpha$, $T_j(\alpha)$, to be the sum

$$T_j(\alpha) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} a_{i_1} a_{i_2} \cdots a_{i_j} \pmod p.$$

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These are the elementary symmetric functions evaluated at $a_1, a_2, \ldots, a_n$. Clearly, $(-1)^j T_j(\alpha)$ is the negation of the coefficient of $z^{n-j}$ in the polynomial

$$(z - a_1)(z - a_2) \cdots (z - a_n).$$

By $S_p(n; \tau_1, \tau_2, \ldots, \tau_j)$ we denote the number of strings $\alpha$ over $R$ of length $n$ for which $T_i(\alpha) = \tau_i$ for $i = 1, 2, \ldots, j$. Obviously if $j = 0$, then $S_p(n) = r^n$. It is also true that $S_p(n; t) = p^{n-1}$ for any $t \in R$, since $T_1(\alpha x)$ takes on distinct values for each $x \in R$.

In what follows, the notation $[\![P]\!]$ for proposition $P$ has the value 1 if $P$ is true and the value 0 if $P$ is false. This is "Iverson's convention," as used in [3].

The numbers $S_p(n; \tau_1, \tau_2, \ldots, \tau_t)$ satisfy the following recurrence relation. If $n = 1$, then $S_p(n; \tau_1, \tau_2, \ldots, \tau_j) = [\![\tau_2 = \cdots = \tau_j = 0]\!]$, and for $n > 0$,

$$S_p(n; \tau_1, \tau_2, \ldots, \tau_j) = \sum_{x \in \mathbb{Z}_p} S_p(n-1; \rho_1, \rho_2, \ldots, \rho_j), \tag{2.1}$$

where $\rho_0 = 1$, and for $i = 1, 2, \ldots, j$,

$$\rho_i = \tau_i - \rho_{i-1} x.$$

Iterating yields (with $\tau_0 = 1$)

$$\rho_i = \sum_{\ell=0}^{i} (-1)^\ell \tau_{i-\ell} x^\ell.$$

Recurrence relation (2.1) implies that the power series $\sum_{n \geq 0} S_p(n; \tau_1, \tau_2, \ldots, \tau_j) z^n$ is rational. We can evaluate $S_p(n; \tau_1, \tau_2, \ldots, \tau_j)$ by creating a table of size $np^j$ consisting of $S_p$ for all strings of length at most $n$ and over all $j$-traces. Each table entry requires $\Theta(pj)$ ring operations and $\Theta(p)$ arithmetic operations, for a total of $\Theta(njp^{j+1})$ ring operations and $\Theta(np^{j+1})$ arithmetic operations. An aim of this paper is to reduce the number of ring and arithmetic operations required to evaluate $S_p$. We begin in the next subsection by classifying the strings according to the frequency with which particular characters occur.

**2.1. Profiles.** Suppose that the string $\alpha$ has $k_x$ occurrences of the symbol $x$ for $x \in \mathbb{Z}_p$. We refer to the $(p-1)$-tuple of natural numbers $\mathbf{k} = \langle k_1, k_2, \ldots, k_{p-1} \rangle$ as the *profile* of the string. Note that $k_0$ is omitted since it doesn't affect $T_j$. Subsequently, a bold letter will only denote a profile. We add profiles componentwise and define $r\mathbf{k} = \langle rk_1, rk_2, \ldots, rk_n \rangle$ for $r \in \mathbb{Z}_p$.

The $j$-trace $T_j$ depends only on the profile and we have

$$T_j(\alpha) \;=\; \sum_{\substack{\nu_1 + \nu_2 + \cdots + \nu_{r-1} = j \\ 0 \leq \nu_i \leq k_i}} \prod_{i=1}^{r-1} i^{\nu_i} \binom{k_i}{\nu_i} \pmod{p} \tag{2.2}$$

For $\mathbf{k} = \langle k_1, k_2, \ldots, k_{p-1} \rangle \in \mathbb{Z}^{p-1}$, define in $\mathbb{Z}_p[[z]]$ the formal power series

$$A_{\mathbf{k}}(z) = \prod_{j=1}^{p-1} (1 + jz)^{k_j} \tag{2.3}$$

We make no assumption here that the $k_i$ are positive.

Observe that

$$T_j(\alpha) = [z^j]A_{\mathbf{k}}(z), \tag{2.4}$$

where the notation $[z^j]A(z)$ means the coefficient of $z^j$ in the generating function $A(z)$.

LEMMA 2.1.

$$A_{\mathbf{a}+\mathbf{b}}(z) = A_{\mathbf{a}}(z)A_{\mathbf{b}}(z) \tag{2.5}$$

*Proof.* Clear. ∎

Throughout the rest of the paper we assume that $p$ is prime and set $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}_{p^n} = \mathbb{Z}/p^n\mathbb{Z}$. We note that the characteristic of both of these rings is $p$.

THEOREM 2.2. *For all $n > 0$,*

$$A_{p^n\mathbf{k}}(z) = A_{\mathbf{k}}(z^{p^n}).$$

*Proof.* Since $p$ is prime and arithmetic is mod $p$ we have $(1+jz)^{p^n} = 1 + j^{p^n}z^{p^n} = 1 + jz^{p^n}$. Thus

$$A_{p^n\mathbf{k}}(z) = \prod_{j=1}^{p-1}(1+jz)^{p^nk_j} = \prod_{j=1}^{p-1}(1+jz^{p^n})^{k_j} = A_{\mathbf{k}}(z^{p^n}).$$

∎

COROLLARY 2.3. *For all $n > 0$,*

$$A_{\mathbf{a}+p^n\mathbf{b}}(z) = A_{\mathbf{a}}(z) \bmod z^{p^n}$$

*Proof.* By Lemma 2.1 $A_{\mathbf{a}+p^n\mathbf{b}}(z) = A_{\mathbf{a}}(z)A_{p^n\mathbf{b}}(z) = A_{\mathbf{a}}(z)A_{\mathbf{b}}(z^{p^n}) = A_{\mathbf{a}}(z) \pmod{z^{p^n}}$. ∎

Notice that this corollary implies that if we are only considering traces $T_j$ with $j < p^n$, then we need only consider values of the profiles taken mod $p^n$.

We also denote the sum in (2.2) by $T_j(\mathbf{k})$ or $T_j(\langle k_1, k_2, \ldots, k_{p-1}\rangle)$ when we wish to emphasize the role of profiles. Let $\alpha$ and $\beta$ be strings over $\mathbb{Z}_p$. The $j$-trace satisfies a natural convolution

$$T_j(\alpha\beta) \;=\; \sum_{0\le i\le j} T_i(\alpha)T_{j-i}(\beta) \tag{2.6}$$

In terms of profiles, this becomes

$$T_j(\mathbf{k}+\mathbf{k}') \;=\; \sum_{0\le i\le j} T_i(\mathbf{k})T_{j-i}(\mathbf{k}'). \tag{2.7}$$

The evaluation of $S_p$ in terms of profiles is given below.

$$S_p(n;\tau_1,\tau_2,\ldots,\tau_t) = \sum_{\substack{k_0+k_1+\cdots+k_{p-1}=n\\ \mathbf{k}:=\langle k_1,\ldots,k_{p-1}\rangle}} \binom{n}{k_0,k_1,\ldots,k_{p-1}} \prod_{i=1}^{t} [\![T_i(\mathbf{k}) = \tau_i]\!] \tag{2.8}$$

In order to evaluate (2.8) efficiently we need to be able to determine efficiently those profiles $\mathbf{k}$ for which $T_i(\mathbf{k}) = \rho_i$ for $i = 1, 2, \ldots, t$. We will do this in the sections to follow.

## 3. The Rings $\mathbb{Z}_p$ and $\mathbb{Z}_{p^n}$.

**3.1. The fundamental correspondence.** We first show that the map $f$ that sends the $(p-1)$-tuple $\mathbf{k} = \langle k_1, k_2, \ldots, k_{p-1}\rangle$ to $\langle \tau_1, \tau_2, \ldots, \tau_{p-1}\rangle$, where $\tau_j = T_j(\mathbf{k}) = \sum \prod i^{v_i}\binom{k_i}{\nu_i}$, is a bijection on $\mathbb{Z}_p^{p-1}$.

LEMMA 3.1. *Let $p$ be a prime and $V$ be the $(p-1)\times(p-1)$ Vandermonde matrix defined by $v_{i,j} = j^i \pmod p$. Then $V^{-1} = W$ is the $(p-1)\times(p-1)$ matrix defined by $w_{i,j} = -i^{-j} \pmod p$.* PROOF. Let $c_{i,j}$ be the $i,j$ entry of the matrix product $VW$.

$$c_{i,j} = -\sum_{1\le k<p} k^{i-j} = \begin{cases} 0 & \text{if } i \neq j \\ -(p-1) & \text{if } i = j \end{cases}$$

Thus $c_{i,j} = [\![ i = j \bmod p ]\!]$. The second equality follows from the proof of the first theorem about characters on finite abelian groups as applied to the map $\chi : x \mapsto x^{i-j}$ on $\mathbb{Z}_p{}^* = \mathbb{Z}_p \setminus \{0\}$. That is, $\sum_{g\in G}\chi(g) = |G| \cdot [\![\chi \text{ is trivial}]\!]$ (e.g., [4], Theorem 5.4). ☐

Clearly $f$ is a function; we will prove that it has an inverse $f^{-1}$. We refer to the result of the following theorem as the "Fundamental Correspondence."

THEOREM 3.2. *The map $f : \mathbb{Z}_p^{p-1} \mapsto \mathbb{Z}_p^{p-1}$ defined by $f(\mathbf{k}) = \langle \tau_1, \tau_2, \ldots, \tau_{p-1}\rangle$, where $\tau_i = T_i(\mathbf{k})$, is a bijection. Both $f$ and $f^{-1}$ can be computed in $O(p^2)$ arithmetic operations.*

PROOF: The $j$th power symmetric function in variables $x_1, x_2, \ldots, x_t$, denoted $P_j(x_1, \ldots, x_t)$, is defined as

$$P_j(x_1, x_2, \ldots, x_t) = \sum_{i=1}^{t} x_i^j.$$

The Newton-Girard Formula

$$mT_m(x_1, x_2, \ldots, x_t) + \sum_{1\le j\le m} (-1)^j P_j(x_1, x_2, \ldots, x_t) T_{m-j}(x_1, x_2, \ldots, x_t) = 0 \tag{3.1}$$

allows us to express a power symmetric function as a (unique) polynomial of elementary symmetric functions. Given fixed values of the variables, $P_m = P_m(x_1, x_2, \ldots, x_t)$ and $T_m = T_m(x_1, x_2, \ldots, x_t)$ are values, and we can use (3.1) to compute unique values $P_1, P_2, \ldots, P_r$ from $T_1, T_2, \ldots, T_r$ by iterating the following equation for $m = 1, 2, \ldots, r$ (in that order). The successive computation of $P_1, P_2, \ldots, P_r$ will clearly take a total of $\Theta(p^2)$ arithmetic steps.

$$P_m = (-1)^{m+1}\left( mT_m + \sum_{1\le j\le m-1} (-1)^j P_j T_{m-j} \right)$$

Note that, as a function of the profile, $P_j = P_j(\langle k_1, k_2, \ldots, k_{p-1}\rangle) = \sum_{i=1}^{p-1} k_i i^j$. We therefore have the system of linear equations $\langle P_1, P_2, \ldots, P_{p-1}\rangle^T = V_p\langle k_1, k_2, \ldots, k_{p-1}\rangle^T$, where $V_p$ is the $(p-1)\times(p-1)$ Vandermonde matrix with $V_p[i,j] = j^i \bmod p$. Since the Vandermonde matrix is non-singular, this system has a unique solution $\langle k_1, k_2, \ldots, k_{p-1}\rangle$, thereby showing that $f^{-1}$ is a function, as claimed. Further, the explicit expression for $V_p^{-1}$ given in Lemma 3.1, allows us to compute the profile in $\Theta(p^2)$ arithmetic operations in $\mathbb{Z}_p$. ☐

The corollary below follows at once from Theorem 3.2 and the observation that $T_i(0, 0, \ldots, 0) = 0$ for any $i > 0$.

COROLLARY 3.3. *If $T_i(\mathbf{k} \bmod p) = 0$ for $i = 1, 2, \ldots, p-1$, then $k_1 = k_2 = \cdots = k_{p-1} = 0$.*

**Example 1:**

Let us determine, in $\mathbb{Z}_7$, the profile that corresponds to the trace values $(T_1, T_2, \ldots, T_{p-1}) = (1, 1, 1, 1, 1, 1)$. The Newton-Girard Formula can be written as

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ P_1 & -2 & & & & \\ P_2 & -P_1 & 3 & & & \\ P_3 & -P_2 & P_1 & -4 & & \\ P_4 & -P_3 & P_2 & -P_1 & 5 & \\ P_5 & -P_4 & P_3 & -P_2 & P_1 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solving by back substitution we get $(P_1, P_2, P_3, P_4, P_5, P_6)^T = (1, 6, 1, 6, 1, 6)^T$.

We now solve $(1, 6, 1, 6, 1, 6)^T = V_7\langle k_1, k_2, \ldots, k_{p-1}\rangle$,

$$\begin{bmatrix} 1 \\ 6 \\ 1 \\ 6 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 2 & 4 & 1 \\ 1 & 1 & 6 & 1 & 6 & 6 \\ 1 & 2 & 4 & 4 & 2 & 1 \\ 1 & 4 & 5 & 2 & 3 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{bmatrix},$$

by using the inverse $V_7^{-1}$ computed from Lemma 3.1

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 6 & 6 & 6 & 6 \\ 3 & 5 & 6 & 3 & 5 & 6 \\ 2 & 3 & 1 & 5 & 4 & 6 \\ 5 & 3 & 6 & 5 & 3 & 6 \\ 4 & 5 & 1 & 3 & 2 & 6 \\ 1 & 6 & 1 & 6 & 1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 1 \\ 6 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 6 \end{bmatrix}$$

Thus the number of strings $\alpha$ of length $n$ over $\mathbb{Z}_7$ with $T_j(\alpha) = 1$ for $j = 1, 2, 3, 4, 5, 6$ is

$$S_{\mathbb{Z}_7}(n; 1, 1, 1, 1, 1, 1) = \sum_{\substack{k_0+k_1+\cdots+k_6=n \\ k_1\equiv\cdots\equiv k_5\equiv 0 \wedge k_6\equiv 6 (\bmod 7)}} \binom{n}{k_0, k_1, \ldots, k_6} \tag{3.2}$$

The actual values for $n = 1, 2, \ldots, 20$ are 0, 0, 0, 0, 0, 1, 7, 28, 84, 210, 462, 924, 10297, 123137, 906010, 4813368, 20435156, 73540572, 232846824, 1996062481. This computation takes a couple of seconds in Maple, after re-arranging (3.2) into the form

$$\sum_{m=0}^{\lfloor (n-6)/7 \rfloor} \binom{n}{7m+6} \sum_{\substack{\nu_1+\cdots+\nu_6=m \\ \nu_i \geq 0}} \binom{7m+6}{7\nu_1, \ldots, 7\nu_5, 7\nu_6+6}.$$

Note that the number of terms in the above sum is about $\binom{n/7+6}{7}$.

Using classical results about primitive roots of unity (see the Appendix) we can express (3.2) as a sum of $7^6$ terms, each term of which raises a complex number to the power $n$. Equation (3.2) can be written as

$$\frac{1}{7^6} \sum_{\nu_1=0}^{6} \cdots \sum_{\nu_6=0}^{6} \omega^{\nu_6}(1 + \omega^{\nu_1} + \cdots + \omega^{\nu_6})^n, \tag{3.3}$$

where $\omega$ is a primitive 7-th root of unity. In infinite precision complex arithmetic we can evaluate sums such as (3.3) in time $\Theta(p^{p-1} \log n)$ by using binary powering. However, in Maple the computation using (3.3) is much slower for realistic values of $n$.

**3.2. Extending the fundamental correspondence.** In this subsection we will prove that the map $f_n : \mathbb{Z}_{p^n}^{p-1} \mapsto \mathbb{Z}_p^{p^n-1}$ that sends $\mathbf{k} = \langle k_1, k_2, \ldots k_{p-1}\rangle \bmod p^n$ to $\langle \tau_1, \tau_2, \ldots, \tau_{p^n-1}\rangle \bmod p$, where $\tau_j = T_j(\mathbf{k})$, is one-to-one and determine its range for all $n \geq 2$. Let $\mathcal{P}_j = \{p^j, 2p^j, \ldots, (p-1)p^j\}$. We call the union $\mathcal{R}_m = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_m$ the *critical set* for the sequence $T_1, T_2, \ldots, T_{p^m-1}$; the elements of $\mathcal{R}_m$ are called *critical indices*. In extending the fundamental correspondence we will prove that the map $f_m$, restricted to the values $T_j$ where $j \in \mathcal{R}_m$, is a bijection. The values of $T_j$ where $j$ is not critical are determined by the values of $T_i$ on the critical indices $i < j$. In the previous subsection we showed that $f_1$ is a bijection on $\mathbb{Z}_p^{p-1}$.

$$\underbrace{\underbrace{\underbrace{\overbrace{1,2,\ldots,p-1}^{\mathcal{P}_1,\text{ taken mod }p}}_{f_1}, \overbrace{p,\ldots,2p,\ldots,(p-1)p}^{\mathcal{P}_2,\text{ taken mod }p^2}}_{f_2}, \ldots, \overbrace{p^{m-1},\ldots,2p^{m-1},\ldots,(p-1)p^{m-1},\ldots}^{\mathcal{P}_m,\text{ taken mod }p^m}}_{f_m}, p^m$$

Lemma 3.4. *$A_{\mathbf{a}}(z) = A_{\mathbf{b}}(z) \bmod z^{p^n}$ if and only if $\mathbf{a} \equiv \mathbf{b} \bmod p^n$.*

*Proof.* If $\mathbf{a} \equiv \mathbf{b} \bmod p^n$ then by Corollary 2.3 $A_{\mathbf{a}}(z) = A_{\mathbf{b}}(z) \pmod{z^{p^n}}$.

Conversely, assume that $A_{\mathbf{a}}(z) = A_{\mathbf{b}}(z) \bmod z^{p^n}$. Then by (2.5) $A_{\mathbf{a}-\mathbf{b}}(z) = 1 \bmod z^{p^n}$. We proceed by induction on $n$. If $n = 1$ then by the Fundamental Correspondence $\mathbf{a} \equiv \mathbf{b} \bmod p$. If $n > 1$ then we may assume inductively that $\mathbf{a} \equiv \mathbf{b} \bmod p^{n-1}$. Thus there is some $\mathbf{k} \in \mathbb{Z}_p^{p-1}$ such that $\mathbf{a} = \mathbf{b} + p^{n-1}\mathbf{k}$ and thus

$$1 = A_{\mathbf{a}-\mathbf{b}}(z) = A_{p^{n-1}\mathbf{k}}(z) = A_{\mathbf{k}}(z^{p^{n-1}}) \pmod{z^{p^n}}.$$

Since the condition $1 = A_{\mathbf{k}}(z^{p^{n-1}}) \bmod z^{p^n}$ is equivalent to $1 = A_{\mathbf{k}}(z) \bmod z^p$, again applying the Fundamental Correspondence, $\mathbf{k} = \mathbf{0} \bmod p$. ▯

Theorem 3.5. *The function $f_n$ is one-to-one.*

*Proof.* We now assume that $\mathbf{k} \in \mathbb{Z}_{p^n}^{p-1}$. Lemma 3.4 shows that $f_n$ (i.e., $A_{\mathbf{k}}(z) \bmod z^{p^n}$ regarded as a function of $\mathbf{k}$) is an injection. ▯

**3.3. The range of $f_n$.** Theorem 3.6. *The range of $f_n : \mathbb{Z}_{p^n}^{p-1} \mapsto \mathbb{Z}_p^{p^n-1}$ consists of all vectors*

$$\langle a_1, \ldots, a_{p^{n-1}-1}, a_{p^{n-1}}, \ldots, a_{2p^{n-1}}, \ldots, a_{(p-1)p^{n-1}}, \ldots, a_{p^n-1}\rangle$$

*where*

*(i) The vector $\langle a_1, \ldots, a_{p^{n-1}-1}\rangle \in$ Range$(f_{n-1})$.*
*(ii) The values of $a_{mp^{n-1}}$ can be assigned arbitrarily from $\mathbb{Z}_p$ for $m = 1, 2, \ldots, p-1$.*
*(iii) For such an assignment, there are unique vectors $\mathbf{a} \in \mathbb{Z}_p^{p-1}$ and $\mathbf{b} \in \mathbb{Z}_{p^{n-1}}^{p-1}$ such that $T_j(\mathbf{b}) = a_j$ for $j = 1, 2, \ldots, p^{n-1} - 1$ and $T_{mp^{n-1}}(p^{n-1}\mathbf{a} + \mathbf{b}) = a_{mp^{n-1}}$ for $m = 1, 2, \ldots, p-1$.*
*(iv) The $a_j$ for $(m-1)p^{n-1} < j < mp^{n-1}$ where $1 < m \leq p$ are determined uniquely as $a_j = T_j(p^{n-1}\mathbf{a} + \mathbf{b})$.*

*Proof.* Our proof is by induction on $n$. Given $(a_1, \ldots, a_{p^{n-1}-1}) \in \mathrm{Range}(f_{n-1})$ there is a unique vector $\mathbf{b} \in \mathbb{Z}_{p^{n-1}}^{p-1}$ such that $[z^\ell]A_{\mathbf{b}}(z) = T_\ell(\mathbf{b}) = a_\ell$ for $\ell = 1, 2, \ldots, p^{n-1} - 1$. Write $\mathbf{k} = p^{n-1}\mathbf{a} + \mathbf{b}$. If $1 \leq \ell < p^{n-1}$ then $[z^\ell]A_{\mathbf{k}}(z) = [z^\ell](A_{p^{n-1}\mathbf{a}+\mathbf{b}}(z) \bmod p^{n-1}) = [z^\ell]A_{\mathbf{b}}(z)$.

$$\begin{aligned}[z^{mp^{n-1}}]A_{\mathbf{k}}(z) &= [z^{mp^{n-1}}]A_{p^{n-1}\mathbf{a}+\mathbf{b}}(z)\\ &= [z^{mp^{n-1}}]A_{\mathbf{a}}(z^{p^{n-1}})A_{\mathbf{b}}(z)\\ &= \sum_{0\leq j\leq m} T_{p^{n-1}j}(\mathbf{b})T_{m-j}(\mathbf{a}).\end{aligned}$$

Thus we are led to consider the equations

$$a_{mp^{n-1}} = \sum_{0\le j\le m} T_{p^{n-1}j}(\mathbf{b})x_{m-j}$$

where $x_j = T_j(\mathbf{a})$. With $x_0 = 1$, we can uniquely determine the values $x_1, x_2, \ldots, x_{p-1}$ successively by substitution in

$$x_m = a_{mp^{n-1}} - \sum_{1\le j\le m} T_{p^{n-1}j}(\mathbf{b})x_{m-j}.$$

By the Fundamental Correspondence, the equations $x_j = T_j(\mathbf{a})$ for $j = 1, 2, \ldots, p-1$ have a unique solution $\mathbf{a}$. Thus $\mathbf{k} = p^{n-1}\mathbf{a}+\mathbf{b}$ is a profile for which $a_i = T_i(\mathbf{k})$ for all $i \in \mathcal{R}_n = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_n$. There are exactly $p^{n(p-1)}$ profiles of the form $p^{n-1}\mathbf{a} + \mathbf{b}$ and exactly $p^{n(p-1)}$ tuples $a_i$ for $i \in \mathcal{R}_n$. Therefore, $f_n$ is a bijection when restricted to $\mathcal{R}_n$. Furthermore, the values $a_j$ for $j \in \{1, 2, \ldots, p^n - 1\} \setminus \mathcal{R}_n$ are uniquely determined as $a_j = T_j(p^{n-1}\mathbf{a} + \mathbf{b})$. □

We showed above that the trace values are determined by the values of traces whose indices are in the critical set. We refine this below by showing that that the value of $T_t$ for $t$ non-critical depends only on the values of $T_j$ where $j < t$ and $j$ is critical.

Theorem 3.7. *The value of $T_t(\alpha)$ where $mp^{n-1} < t < (m+1)p^{n-1}$ is determined by the values of $\tau_j = T_j(\alpha)$ for $j \in \mathcal{R}_{n-1} \cup \{p^{n-1}, 2p^{n-1}, \ldots, mp^{n-1}\}$.*

*Proof.* By our previous results on the range of $f_n$, we know that there are exactly $p^{p-1-m}$ profiles $\mathbf{k}$ such that $T_j(\mathbf{k}) = \tau_j$ for $j \in \mathcal{R}_{n-1} \cup \{p^{n-1}, 2p^{n-1}, \ldots, mp^{n-1}\}$. Such a profile $\mathbf{k}$ can be written as $p^{n-1}\mathbf{a} + \mathbf{b}$, where $\mathbf{a} \in \mathbb{Z}_p^{p-1}$ and $\mathbf{b} \in \mathbb{Z}_{p^{n-1}}^{p-1}$.

Consider profiles $\mathbf{k} = p^{n-1}\mathbf{a}+\mathbf{b}$ and $\mathbf{k}' = p^{n-1}\mathbf{a}'+\mathbf{b}'$ where $T_j = \tau_j$ for $j \in \mathcal{R}_{n-1}\cup\{p^{n-1}, 2p^{n-1}, \ldots, mp^{n-1}\}$. Since the profiles agree on $\mathcal{R}_{n-1}$ we have $A_{\mathbf{k}}(z) = A_{\mathbf{k}'}(z) \bmod z^{p^{n-1}}$ and hence $\mathbf{b} = \mathbf{b}'$. Since, for $j = 1, 2, \ldots, m$,

$$[z^{jp^{n-1}}]A_{\mathbf{k}}(z) = [z^{jp^{n-1}}]A_{\mathbf{k}'}(z)$$

we have, for $j = 1, 2, \ldots, m$,

$$[z^{jp^{n-1}}]A_{\mathbf{a}}(z^{p^{n-1}})A_{\mathbf{b}}(z) = [z^{jp^{n-1}}]A_{\mathbf{a}'}(z^{p^{n-1}})A_{\mathbf{b}}(z).$$

Which implies that

$$A_{\mathbf{a}}(z) = A_{\mathbf{a}'}(z) \pmod{z^{m+1}}.$$

Now note that

$$\begin{aligned}
[z^t]A_{\mathbf{k}}(z) - [z^t]A_{\mathbf{k}}(z) &= [z^t](A_{\mathbf{a}}(z^{p^{n-1}}) - A_{\mathbf{a}'}(z^{p^{n-1}}))A_{\mathbf{b}}(z) \\
&= [z^t](A_{\mathbf{a}}(z^{p^{n-1}}) - A_{\mathbf{a}'}(z^{p^{n-1}}))A_{\mathbf{b}}(z) \pmod{z^{(m+1)p^{n-1}}} \\
&= 0
\end{aligned}$$

□

**3.4. A computational method and examples.** In this subsection we give an explicit algorithm in the form of pseudo-code to determine if $\tau_1, \tau_2, \ldots, \tau_{p^n-1} \in \mathrm{Range}(f_n)$, and, if so, how to find the profile $\mathbf{p} \in \mathbb{Z}_{p^n}^{p-1}$ such that $T_j(\mathbf{p}) = \tau_j$ for $j = 1, 2, \ldots, p^n - 1$. In particular, we will determine a sequence $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{n-1}$, where each $\mathbf{a}_i \in \mathbb{Z}_p^{p-1}$, such that

$$\mathbf{p} = \mathbf{a}_0 + p\mathbf{a}_1 + \cdots + p^{n-1}\mathbf{a}_{n-1}.$$

The principles underlying the algorithm have already been laid out in Theorem 3.2 and Theorem 3.6.

(A1) $\mathbf{a} := \mathbf{0}$; $x_0 := 1$;
(A2) **for** $i := 0$ **to** $n-1$ **do**
(A3) **for** $j := 1$ **to** $p-1$ **do**
(A4) $x_j := \tau_{jp^i} - \sum_{i=1}^{j} T_{jp^i}(\mathbf{a}) x_{j-i}$;
(A5) **for** $m := 1$ **to** $p-1$ **do** { Newton-Girard }
(A6) $P_m := (-1)^{m+1} \left( m x_m + \sum_{1 \le j \le m-1} (-1)^j P_j x_{m-j} \right)$;
(A7) **for** $j := 1$ **to** $p-1$ **do** { inverse of Vandermonde }
(A8) $a_j := \sum_{1 \le i \le p-1} (-1)^{j+1} (p-i)^{p-j-1} P_i$;
(A9) $\mathbf{a}_i := \mathbf{a}$;
(A10) $\mathbf{p} := \mathbf{a}_0 + p\mathbf{a}_1 + \cdots + p^{n-1}\mathbf{a}_{n-1}$;
(A11) **for** $i$ **:= 1 to** $p^n - 1$ **do**
(A12) **if** $i \notin \mathcal{R}_n$ **and** $T_i(\mathbf{p}) \neq \tau_i$ **then** return( “no profile exists” );
(A13) return( $\mathbf{p}$ );

**Example 2:**

Let us determine, in $\mathbb{Z}_{p^2}$, the profile $\mathbf{p}$, if any, that corresponds to the trace values $(T_1, T_2, \ldots, T_{p^2-1}) = (1, 1, \ldots, 1)$, with $p = 7$.

The $i = 0$ iteration of the algorithm was done in the Example 1; $\mathbf{a} = \mathbf{a}_0 = (0,0,0,0,0,6)$. For $i = 1$, the repeated substitution of lines A3–A4 yields (with $\mathbf{b} = \mathbf{a}_1$)

$$(T_1(\mathbf{b}), T_2(\mathbf{b}), T_3(\mathbf{b}), T_4(\mathbf{b}), T_5(\mathbf{b}), T_6(\mathbf{b})) = (x_1, x_2, x_3, x_4, x_5, x_6) = (1,1,1,1,1,1),$$

which is solved (lines A5–A8) as in the previous example to give $(b_1, b_2, b_3, b_4, b_5, b_6) = (0,0,0,0,0,6)$. Thus $\mathbf{p} = 7\mathbf{b} + \mathbf{a} = (0,0,0,0,0,48)$, where $48 = 7 \cdot b_6 + a_6 = 7{\cdot}6 + 6$. We now need to check at lines A11–A12 whether $T_j(\mathbf{p}) = 1$ for $7(m-1) < j < 7m$ for $m = 2,3,4,5,6,7$. Consider a string of 48 6's. Clearly,

$$T_j(\mathbf{p}) = 6^j \binom{48}{j} = (-1)^j (-1)^j = 1 \pmod{7},$$

so long as $j \le 48$. (To see that $\binom{48}{j} \equiv (-1)^j$ argue by induction using the recurrence relation $0 \equiv \binom{7^2}{j} = \binom{48}{j} + \binom{48}{j-1}$.) In terms of generating functions $(1-z)^{48} = 1 + z + \cdots + z^{48} \bmod 7$. Thus the $T_j$ values are indeed all 1, and we can therefore determine that the number of strings of length $n$ whose first 48 traces are all 1's is

$$S_{\mathbb{Z}_7}(n; \underbrace{1, 1, \ldots, 1}_{48}) = \sum_{\substack{k_0 + k_1 + \cdots + k_6 = n \\ k_1 \equiv \cdots \equiv k_5 \equiv 0 \wedge k_6 \equiv 48 \pmod{7^2}}} \binom{n}{k_0, k_1, \ldots, k_6}. \tag{3.4}$$

Note that $7^{48} > 3 \times 10^{40}$ so there is no hope of using the recurrence relation (2.1) for the computation.

**Example 3:**

Going in the other direction, specifying fewer traces to be 1, we show how to determine $S_{\mathbb{Z}_7}(n; 1, 1, 1)$. Since we don't have the complete set $\mathcal{P}_1$ we do not have a one-to-one correspondence. However, we can sum $S_{\mathbb{Z}_7}(n; 1, 1, 1, x, y, z)$ over all $x, y, z \in \mathbb{Z}_7$ to determine our answer. Feeding $(1, 1, 1, x, y, z)$ through the Newton-Girard and Vandermonde formulae gives us:

$$\begin{aligned}
k_1 &\equiv 3 + 2x + 3y + 6z \pmod{7} \\
k_2 &\equiv 5 + 5x + 5y + 6z \pmod{7} \\
k_3 &\equiv 6 + 2x + 6z \pmod{7} \\
k_4 &\equiv 6 + 2y + 6z \pmod{7}
\end{aligned}$$

$$k_5 \equiv 5 + 6x + 4y + 6z \pmod 7$$
$$k_6 \equiv 2 + 6x + 6y + 6z \pmod 7$$

These equations can in turn be used to eliminate $x, y, z$, obtaining:

$$k_4 \equiv k_1 + 4k2 + 3k3 \pmod 7,$$
$$k_5 \equiv 3k_1 + 6k_2 + 6k_3 \pmod 7,$$
$$k_6 \equiv 6 + 6k_1 + 6k_2 + 3k3 \pmod 7.$$

This gives us the equation

$$S_{\mathbb{Z}_7}(n; 1, 1, 1) = \sum_{\substack{k_0+k_1+\cdots+k_6=n \\ k_4 \equiv k_1+4k2+3k3 \pmod 7 \\ k_5 \equiv 3k_1+6k_2+6k_3 \pmod 7 \\ k_6 \equiv 6+6k_1+6k_2+3k_3 \pmod 7}} \binom{n}{k_0, k_1, \ldots, k_6}.$$

**Example 4:**

As another example we will determine a formula for the number of binary strings of length $n$ whose first $2^m$ traces are all 0's. I.e., we will determine the number

$$A(n, m) := S_{\mathbb{Z}_2}(n; \underbrace{0, 0, \ldots, 0}_{2^m})$$

According to Theorem 3.6 the relevant trace values are $T_j$ for $j = 1, 2, \ldots, 2^m$. From (.1) of the Appendix,

$$A(n, m) = \sum_{\substack{j \geq 0 \\ j \equiv 0 \pmod{2^m}}} \binom{n}{j} = \frac{1}{2^m} \sum_{j=0}^{2^m-1} (1 + \omega^j)^n = \frac{1}{2^m} \sum_{j=0}^{2^m-1} \left(2 \cos \frac{\pi j}{2^m}\right)^n \cos \frac{\pi j n}{2^m},$$

where $\omega$ is a primitive $2^m$-th root of unity. The last equality follows from the observation that $(1 + \omega^j) = \omega^{j/2}(\omega^{-j/2} + \omega^{j/2})$.

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**4. Appendix.** Roots of Unity

For fixed $n$, it is well know that the sum of every other binomial coefficient is $2^{n-1}$. But what about sums of every $k$-th binomial coefficient? What about similar sums of multinomial coefficients? It turns out that we can derive formulae for these whose computation is more efficient than directly summing the coefficients. We start with binomial coefficients, and then proceed to the multinomial coefficients.

Let $\omega$ be a primitive $q$-th root of unity, say $\omega = e^{2\pi i/q}$. Consider the geometric sum below for $q \nmid n$.

$$\sum_{k=0}^{q-1} \omega^{nk} = \frac{1-\omega^{qn}}{1-\omega^{n}} = 0$$

On the other hand if $q \mid n$, then $\omega^{nk} = 1$. Thus

$$\frac{1}{q}\sum_{k=0}^{q-1} \omega^{nk} = [\![ q \mid n ]\!]$$

Let $A(z) = \sum_{n\geq 0} f(n) z^n$. We wish to find an expression for the related generating function that picks off every $q$-th element, starting with the $r$-th element $(0 \leq r < q)$.

$$A_{q;r}(z) = \sum_{n\geq 0} f(nq+r) z^{nq+r}.$$

Set $m = nq + r$. Note that

$$\begin{aligned} A_{q;r}(z) &= \sum_{m\geq 0} [\![ q \mid (m-r) ]\!] f(m) z^m \\ &= \sum_{m\geq 0} \frac{1}{q} \sum_{k=0}^{q-1} \omega^{(m-r)k} f(m) z^m \\ &= \frac{1}{q} \sum_{k=0}^{q-1} \sum_{m\geq 0} f(m) z^m \omega^{mk} \omega^{-rk} \\ &= \frac{1}{q} \sum_{k=0}^{q-1} \omega^{-rk} A(z\omega^k) \end{aligned}$$

An entirely analogous argument in the multidimensional case gives us the following lemma.

LEMMA .1. *Let $A(z_1, z_2, \ldots, z_m)$ be the ordinary generating function of $f(n_1, n_2, \ldots, n_m)$. Define*

$$A_{q;r_1,\ldots,r_m}(z_1, z_2, \ldots, z_m) = \sum_{n_1\geq 0} \cdots \sum_{n_m\geq 0} f(n_1 q + r_1, \ldots, n_m q + r_m) z_1^{n_1 q + r_1} \cdots z_m^{n_m q + r_m}.$$

*where each $r_i \in \mathbb{Z}_q$. Then*

$$A_{q;r_1,\ldots,r_m}(z_1, z_2, \ldots, z_m) = \frac{1}{q^m} \sum_{\nu_1=0}^{q-1} \cdots \sum_{\nu_m=0}^{q-1} \omega^{-(\nu_1 r_1 + \cdots + \nu_m r_m)} A(\omega^{z_1 \nu_1}, \ldots, \omega^{z_m \nu_m})$$

Recall that

$$B(z) = (1+z)^n = \sum_{r=0}^{n} \binom{n}{r} z^r.$$

Substituting $z = 1$ into $B_{q;r}(z)$ we obtain

$$\sum_{j \equiv r(q)} \binom{n}{r} = \frac{1}{q} \sum_{j=0}^{q-1} \omega^{-rj}(1+\omega^j)^n. \tag{.1}$$

Introduce the notation

$$M_q(n; r_1, r_2, \ldots, r_m) = \sum_{\substack{\nu_0+\nu_1+\cdots+\nu_m=n \\ \nu_1 \equiv r_1(q), \ldots, \nu_t \equiv r_m(q)}} \binom{n}{\nu_0, \nu_1, \ldots, \nu_m}$$

Plugging $z_1 = z_2 = \cdots = z_m = 1$ into the ordinary generating function $(1+z_1+\cdots+z_m)^n$ for the multinomial coefficients we obtain the following lemma which generalizes (.1).

Lemma .2. *For all $q \geq 2$, $n \geq 0$, and $r_i \in \mathbb{Z}_q$,*

$$M_q(n; r_1, r_2, \ldots, r_m) = \frac{1}{q^m} \sum_{\nu_1=0}^{q-1} \cdots \sum_{\nu_m=0}^{q-1} \omega^{-(\nu_1 r_1 + \cdots + \nu_m r_m)} (1 + \omega^{\nu_1} + \cdots + \omega^{\nu_m})^n.$$

(Note: in the binomial case, see Knuth vol 1, exercise 38, page 70. He attributes the roots of unity formula to C. Ramus, 1834).