On the Hamiltonicity of Directed $\sigma - \tau$ Cayley Graphs (Or: A Tale of Backtracking)

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Abstract

Let τ be the 2-cycle $(1\ 2)$ and σ the n-cycle $(1\ 2\ \cdots\ n)$. These two cycles generate the symmetric group S_n. Let G_n denote the directed Cayley graph $Cay(\{\tau, \sigma\} : S_n)$. Based on erroneous computer calculations, Nijenhuis and Wilf ([NiWi75], pg. 238 and [NiWi78], pg. 288) give as an exercise to show that G_5 does not have a Hamilton path. To the contrary, we show that G_5 is Hamiltonian. Furthermore, we show that G_6 has a Hamilton path. Our results illustrate how a little theory and some good luck can save a lot of time in backtracking searches.

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1 Introduction

There has been much research interest in Hamiltonicity properties of Cayley Graphs. This interest is fueled by the very attractive conjecture (of Parsons and others) that all connected undirected Cayley graphs are Hamiltonian, and most research has centered on undirected Cayley graphs. In the case of directed Cayley graphs, the question is messier, in part because of the existence of examples of directed Cayley graphs that are not Hamiltonian and there are even examples that don't have Hamilton paths. Among directed Cayley graphs without Hamilton paths, the "sigma-tau" graph for $n = 5$ is often cited (e.g., Nijenhuis and Wilf [NiWi75], [NiWi78], Holsztyński and Strube [HoSt], Wilf [Wilf], Witte [Witt], and Witte and Gallian [WiGa]).

To be precise, let

$$\tau = (1\ 2) \quad \text{and} \quad \sigma = (1\ 2\ \cdots\ n),$$

and let G_n denote the Cayley graph $Cay(\{\sigma, \tau\} : S_n)$. We will show that the sigma-tau examples should not be used because G_4 has a Hamilton path but is not Hamiltonian, G_5 is Hamiltonian, and G_6 has a Hamilton path. Whether G_6 has a Hamilton cycle is not known.

In a recent tour-de-force Compton and Williamson [CoWi] have shown that the undirected Cayley graph $Cay(\{\tau, \sigma, \sigma^{-1}\} : S_n)$ is Hamiltonian for all n. The question of determing the Hamiltonicity of G_n appears to be much more difficult.

Our basic technique in discovering these Hamilton paths and cycles is the time-honored technique of backtracking. The paper is organized into five sections. In this first section we introduce the problem and notation and present a few simple lemmas. The next three sections present our results for $n = 4, 5, 6$. The final section discusses the backtracking techniques that were used.

We regard Cayley graphs as having edges labeled by a generator and Hamilton paths and cycles as consisting of a string of edge labels. Thus, a Hamilton path in G_n is a string of length $n! - 1$ over the alphabet $\{\sigma, \tau\}$.

The next two lemmas help us to prune the backtracking tree.

Lemma 1 *For $n > 2$, if G_n has a Hamiltonian path then that path must contain a sub-path labeled $\sigma\tau\sigma\sigma$.*

Proof. This follows immediately from the identity $(\tau\sigma)^{n-1} = id$. □

Lemma 2 *Any Hamiltonian cycle in G_n does not contain the subpath $\tau\sigma^{n-2}\tau$.*

Proof. Assume that there is a Hamilton cycle C that starts with $\tau\sigma^{n-2}\tau$. Then there is no way to exit the vertex $x = \tau\sigma^{n-1}$. It cannot be entered from $\tau\sigma^{n-2}$, nor can it be exited to $\tau\sigma^n = \tau$. Thus x must both be exited and entered from $x\tau$, which is impossible. $\square$

Although G_n is directed, it shares one nice property with undirected Cayley graphs, which is not true in general for directed Cayley graphs. If A is a string, then by A^R we denote the reversal of A.

Lemma 3 *If A is a Hamilton path (cycle) in G_n, then A^R is also a Hamilton path (cycle) in G_n.*

Proof. Consider the Cayley graph $G_n^{-1} = Cay(\{\tau, \sigma^{-1}\} : S_n)$. Clearly, if A is a path in G_n, then A^R, with σ replaced by σ^{-1}, is a path in G_n^{-1}. We will show that G_n and G_n^{-1} are isomorphic, by an isomorphism that maps τ edges to τ edges and σ edges to σ^{-1} edges; from this the lemma follows immediately. Consider the permutation

$$g = (1\ 2)(3\ n)(4\ n{-}1)\cdots(\lfloor (n{+}3)/2 \rfloor\ \lceil (n{+}3)/2 \rceil).$$

Define the mapping $x \to xg$, where $x \in S_n$. We will show that if (x, xh) is an edge of G_n, then (xg, xhg) is an edge of G_n^{-1}. Either $h = \tau$, or $h = \sigma$. If $h = \tau$, then

$$\tau g \;=\; g\tau \;=\; (1)(2)(3\ n)(4\ n{-}1)\cdots$$

and $(xg, xg\tau) \in E(G_n^{-1})$. If $h = \sigma$ then

$$\sigma g \;=\; g\sigma^{-1} \;=\; (1\ 3)(2)(4\ n)(5\ n{-}1)\cdots$$

and $(xg, xg\sigma^{-1}) \in E(G_n^{-1})$. $\square$

Thus when we speak of "non-isomorphic" Hamilton paths we consider two paths to be isomorphic if one is the reversal of the other. Similarly, two Hamilton cycles are isomorphic if one can be obtained from the other by rotation or reversal.

Lemma 4 *Any Hamilton path in G_n that starts at the permutation $12\cdots n$ and ends at $x_1x_2\cdots x_n$ must satisfy $x_1 = 2$.*

Proof. This is proven for a spanning supergraph of G_n in [JiRu]. $\square$

However, unlike the Faber-Moore graphs of [JiRu], not all permutations of $1, 3, \ldots, n$ seem to be possible ending permutations of Hamilton paths starting at $12\cdots n$.

It might be remarked that the approach of Compton and Williamson [CoWi] will not work in the directed case. Their cycle successively lists the elements of the cosets of $\langle\sigma\rangle$, a coset at a time. That approach fails work in the directed case since $(\tau\sigma^{n-1})^{n-1} = (\tau\sigma^{-1})^{n-1} = id$.

In what follows we use the integer k to denote $\tau\sigma^{k-1}$. To avoid confusion, these integers will be enclosed in square brackets; e.g., $\tau\sigma\sigma\tau\sigma = [32]$.

2 The case $n = 4$

To understand the proof of the following lemma the reader should refer to Figure 1, which shows $Cay(\{\tau, \sigma\} : S_4)$. It is interesting that the graph $Cay(\{\tau, \sigma, \sigma^{-1}\} : S_4)$ is isomorphic to $Cay(\{(12), (23), (34)\} : S_4)$, the "permutohedron" of S_4 (e.g., Figure 1 of [RuSa]). Directed edges in Figure 1 correspond to σ and the undirected (triple lined) edges correspond to τ.

Lemma 5 *The graph G_4 is not Hamiltonian.*

Proof. Assume to the contrary that G_4 has a Hamilton cycle C. We will derive a contradiction. By Lemma 2, C does not contain $[3]\tau$; i.e., the path is a sequence of 2's and 4's. Our proof makes use of the identities $[2]^3 = [4]^3 = [24]^3 = id$ (valid only for $n = 4$). By $[2]^3 = id$, the cycle C must not contain the subpath [224]. By $[4]^3 = id$, the cycle C must not contain the subpath [444]. By $[24]^3 = id$ it must contain the subpath [44]. Hence, C must contain the subpath $[424424] = id$, which is a contradiction since this subpath contains only 20 vertices. $\square$

Lemma 6 *There are exactly 3 non-isomorphic Hamilton paths in G_4, which are listed below.*

$$\tau\sigma\sigma\sigma\tau\sigma\tau\sigma\sigma\sigma\tau\sigma\sigma\sigma\tau\sigma\tau\sigma\sigma\tau\sigma\sigma\sigma = [4244234] \quad (1)$$

$$\tau\sigma\sigma\sigma\tau\sigma\sigma\sigma\tau\sigma\sigma\tau\sigma\tau\sigma\sigma\sigma\tau\sigma\sigma\sigma\tau\sigma = [4432442] \quad (2)$$

$$\sigma\tau\sigma\tau\sigma\sigma\sigma\tau\sigma\sigma\sigma\tau\sigma\tau\sigma\sigma\sigma\tau\sigma\tau\sigma\sigma\sigma = \sigma[2442424] \quad (3)$$

It is interesting to note that if (1) and (2) are considered as circular arrangements of symbols, then one may be obtained from the other by traversing the strings in opposite directions.

3 The case $n = 5$

For $n = 5$ there exist Hamilton cycles and Hamilton paths that are not cycles in G_n. Recall that two cycles are considered isomorphic if one may be obtained from the other by rotation or reversal. We list the lexicographically smallest cycle or path from each equivalence class.

Lemma 7 *There are exactly 5 non-isomorphic Hamilton cycles in G_5, which are listed below. Each cycle contains 36 τ's and 84 σ's (or, more generally, 12 each of 2's, 3's, and 5's).*

$$A_1 = [223252522235523353355335552233225233 5] \quad (4)$$

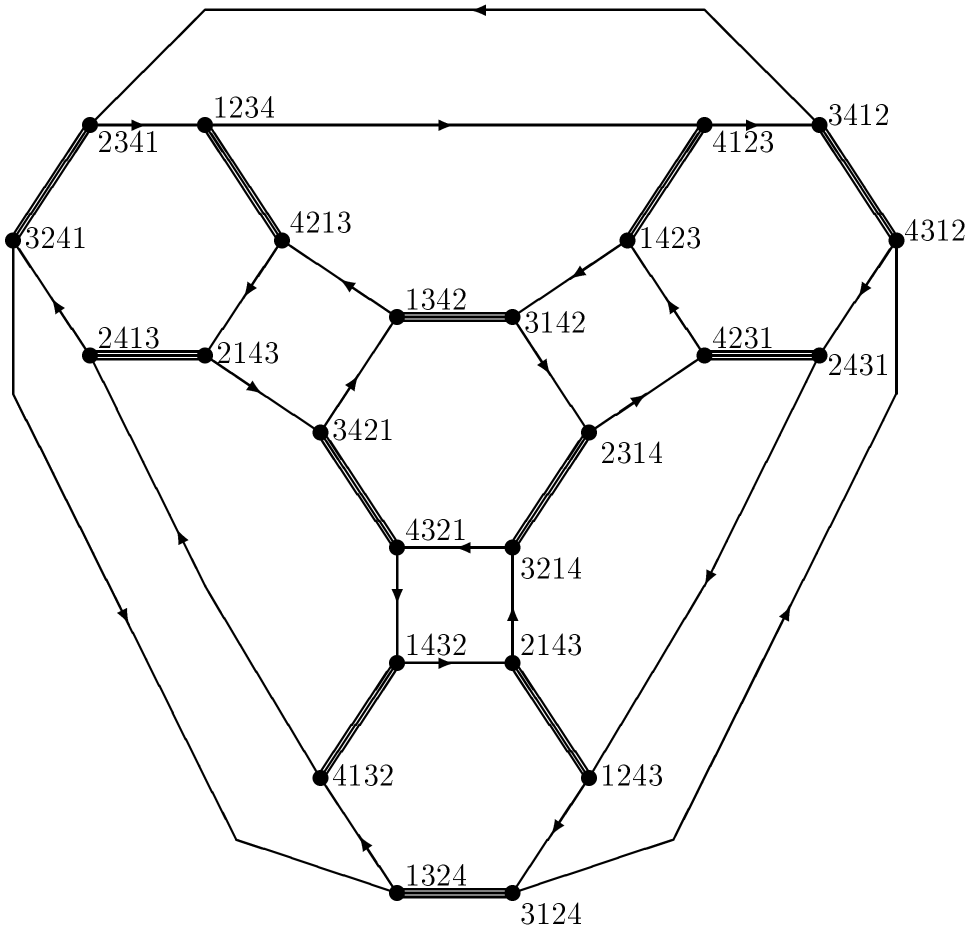

Figure 1: The graph $Cay(\{\tau, \sigma\} : S_4)$.

$$A_2 = [22332352252235523533553522532332553] \tag{5}$$

$$A_3 = [223323522535525335|223323522535525335] \tag{6}$$

$$A_4 = [22332553252|252355233223552353|3532553] \tag{7}$$

$$A_5 = [223533555233235225|223533555233235225] \tag{8}$$

Note that A_3 and A_5 may be written in the form AA. The most beautiful of the cycles is A_4. It has the property that it may be written in the form $A\overline{A}^R\overline{A}A^R$ where $\overline{A}$ is A but with 2's and 3's interchanged. For example, take $A = 252355233$. Now consider the cosets of $\langle\sigma\rangle$; to each coset there is an associated either a [5], or a [2] and a [3]. The [2],[3] cosets of A and $\overline{A}$ are matched as shown below.

$$\begin{array}{ccccccccccc} A & = & 2 & 5 & 2 & 3 & 5 & 5 & 2 & 3 & 3 \\ & & a & & b & c & & & d & e & a \\ & & & & & & & & & & \\ \overline{A} & = & 3 & 5 & 3 & 2 & 5 & 5 & 3 & 2 & 2 \\ & & f & & b & e & & & d & c & f \end{array}$$

The cosets of $\overline{A}^R$ and A^R are matched in a similar manner (but reversed). It is our hope that A_4 may be used as the basis of an inductive proof that G_n is Hamiltonian for all $n \geq 5$.

There also exist Hamilton paths in G_5 which are not Hamilton cycles.

Lemma 8 *There are 430 non-isomorphic Hamilton paths in G_5 that are not Hamilton cycles. Of these, 6 are symmetric in the sense that such a path A satisfies $A = A^R$.*

Here is a symmetric Hamiltonian path in G_5. All 6 symmetric paths had 41 τ's and 78 σ's.

$$[33532233252252332234]\tau[33532233252252332234]^R$$

4 The case $n = 6$

Through our programs we have found that G_6 can be partitioned into two disjoint cycles, one of length 714 and the other of length 6. This cycle was somewhat difficult to find. We experimented with various ratios of τ calls before σ calls and found the 714-cycle by using a 3/7 ratio (recall the 36 to 84 split of Lemma 7).

Lemma 9 *The graph G_6 has a Hamilton path.*

Proof. In Figure 2 is listed a cycle of length 714 in G_6. The six missing vertices, shown below, also form a cycle in G_6.

153246, 246153, 324615, 461532, 532461, 615324

A path may be formed, for example, by taking the six permutations above followed by 165324 in the 714-cycle. □

5 The Backtracking Algorithms

In this section we describe our backtracking implementations. Our overriding concerns were to keep the implementation efficient and as simple as possible.

First a global adjacency list structure was created to represent the graph. With each vertex v four values are stored. Three values, $v\tau$, $v\sigma$, and $v\sigma^{-1}$, are static. The fourth, $visited(v)$, is dynamic and keeps track of whether v is on the path constructed so far, or not.

To obtain the results of the $n = 5$ section we first used the naive backtracking algorithm of Figure 3 which took 99,000 seconds on a Sun MP 690, when carefully implemented in C. We used $gen(\sigma\tau\sigma\sigma)$ as the initial call, which is justified by Lemma 1. We then went through two successive refinements in order to speed-up the algorithm.

Our first refinement of the algorithm took advantage of Lemma 2 and had the form shown in Figure 4. The variable sc is the number of consecutive σ's at the end of A. The initial call is $gen(\sigma\tau\sigma\sigma, 2)$. This simple change had a dramatic effect; it reduced the running time to 850 seconds, but we still felt this was to slow to attack the $n = 6$ problem.

Our second refinement reduced the running time to only 11 seconds. Consider a vertex v of G_n as the backtracking is proceeding. We say that v is *touched* if it is not properly on the cycle constructed so far (e.g., is not a prefix of A), but one of it's neighbors is on the cycle. Observe that if a vertex is touched twice by A, then A cannot be extended to a Hamilton cycle. There are three possible ways that a vertex may be touched, as shown in Figure 5. By using this observation we are led to our second refinement as outlined in Figure 6.

For $n = 6$ we were only able to explore a minute fraction of the backtracking tree. Thus we were unable to determine whether G_6 is Hamiltonian.

6 Acknowledgement

We thank Herb Wilf for his encouragement and his suggestion to look at G_6, which is what got us started on this research.

213456	123456	612345	162345	516234	451623	345162	234516	623451	263451
126345	512634	451263	345126	634512	364512	236451	326451	132645	513264
451326	645132	264513	624513	362451	136245	513624	451362	245136	425136
642513	462513	346251	134625	513462	251346	625134	265134	426513	342651
134265	513426	651342	561342	256134	526134	452613	345261	134526	613452
261345	621345	562134	456213	345621	435621	143562	413562	241356	421356
642135	564213	356421	135642	213564	123564	412356	641235	564123	356412
235641	325641	132564	312564	431256	643125	564312	256431	125643	215643
321564	432156	643215	564321	156432	516432	251643	521643	352164	435216
643521	164352	216435	126435	512643	351264	435126	643512	264351	624351
162435	612435	561243	356124	435612	243561	124356	214356	621435	562143
356214	536214	453621	543621	154362	514362	251436	625143	362514	436251
143625	413625	541362	254136	625413	362541	136254	316254	431625	341625
534162	253416	625341	162534	416253	146253	314625	531462	253146	625314
462531	642531	164253	614253	361425	536142	253614	425361	142536	412536
641253	364125	536412	253641	125364	215364	421536	241536	624153	362415
536241	153624	415362	145362	214536	621453	362145	632145	563214	653214
465321	645321	164532	216453	321645	532164	453216	543216	654321	165432
216543	321654	432165	342165	534216	354216	635421	163542	216354	421635
542163	452163	345216	634521	163452	216345	521634	251634	425163	245163
324516	632451	163245	516324	451632	541632	254163	325416	632541	163254
416325	146325	514632	154632	215463	321546	632154	463215	546321	456321
145632	214563	321456	231456	623145	263145	526314	256314	425631	142563
314256	631425	563142	653142	265314	426531	142653	314265	531426	351426
635142	365142	236514	423651	142365	412365	541236	654123	365412	635412
263541	126354	412635	541263	354126	534126	653412	265341	126534	412653
341265	431265	543126	654312	265431	126543	312654	132654	413265	541326
654132	265413	326541	236541	123654	213654	421365	542136	654213	365421
136542	316542	231654	423165	542316	654231	165423	615423	361542	631542
263154	426315	542631	452631	145263	314526	631452	361452	236145	523614
452361	145236	614523	164523	316452	231645	523164	452316	645231	465231
146523	416523	341652	234165	523416	652341	165234	615234	461523	346152
234615	523461	152346	512346	651234	561234	456123	345612	234561	324561
132456	613245	561324	456132	245613	425613	342561	134256	613425	163425
516342	156342	215634	125634	412563	341256	634125	563412	256341	526341
152634	415263	341526	431526	643152	264315	526431	152643	315264	135264
413526	641352	264135	624135	562413	356241	135624	315624	431562	341562
234156	324156	632415	563241	156324	415632	241563	421563	342156	634215
563421	653421	165342	216534	421653	241653	324165	532416	653241	165324
416532	146532	214653	321465	532146	352146	635214	463521	146352	416352
241635	524163	352416	635241	163524	613524	461352	246135	524613	352461
135246	315246	631524	463152	246315	524631	152463	512463	351246	531246
653124	563124	456312	245631	124563	312456	631245	361245	536124	453612
245361	124536	612453	162453	316245	531624	453162	543162	254316	625431
162543	612543	361254	631254	463125	546312	254631	125463	312546	132546
613254	461325	546132	254613	325461	235461	123546	213546	621354	462135
546213	354621	135462	315462	231546	623154	462315	642315	564231	156423
315642	231564	423156	243156	624315	562431	156243	516243	351624	435162
243516	423516	642351	462351	146235	416235	541623	354162	235416	623541
162354	612354	461235	546123	354612	534612	253461	125346	612534	461253
346125	436125	543612	254361	125436	215436	621543	362154	436215	346215
534621	153462	215346	621534	462153	642153	364215	536421	153642	513642
251364	521364	452136	645213	364521	136452	213645	123645	512364	451236
645123	465123	346512	436512	243651	124365	214365	521436	652143	365214
436521	143652	413652	241365	524136	652413	365241	136524	316524	431652
243165	524316	652431	165243	615243	361524	436152	243615	524361	152436
512436	651243	365124	635124	463512	246351	124635	214635	521463	251463
325146	632514	463251	643251	164325	614325	561432	651432	265143	326514
432651	143265	514326	154326	615432	261543	326154	236154	423615	542361
154236	514236	651423	561423	356142	235614	423561	142356	614235	164235
516423	351642	235164	325164	432516	342516	634251	364251	136425	316425
531642	253164	425316	245316	624531	264531	126453	312645	531264	453126
645312	465312	246531	124653	312465	132465	513246	651324	465132	246513
324651	234651	123465	213465	521346	652134	465213	346521	134652	314652
231465	523146	652314	562314	456231	546231	154623	514623	351462	235146
623514	263514	426351	142635	514263	154263	315426	135426	613542	261354
426135	542613	354261	534261	153426	615342	261534	426153	342615	432615
543261	453261	145326	614532	261453	326145	532614	352614	435261	143526
614352	261435	526143	256143	325614	432561	143256	413256	641325	564132
256413	526413	352641	532641	153264	415326	641532	264153	326415	236415
523641	152364	415236	641523	364152	634152	263415	623415	562341	156234
415623	145623	314562	134562						

Figure 2: A cycle of length 714 in G_6.

```
procedure gen ( A )
begin
    if A is a Hamilton cycle then PrintIt
    else if A is not a cycle then begin
        gen( Aτ );     gen( Aσ );
    end;
end {of gen};
```

Figure 3: Naive Backtracking Algorithm.

```
procedure gen ( A, sc )
begin
    if A is a Hamilton cycle then PrintIt
    else if A is not a cycle then begin
        if sc ≠ n − 2 then gen( Aτ, 0 );
        gen( Aσ, sc + 1 );
    end;
end {of gen};
```

Figure 4: First refinement.

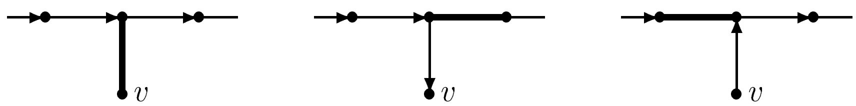

Figure 5: The three ways vertex v can be touched.

```
procedure gen ( A, g )
begin
    if A is a Hamilton cycle then PrintIt
    else if A is not a cycle then begin
        if touched(Aσ) = false then begin
            touched(Aσ) := true;
            gen( Aτ, τ );
            touched(Aσ) := false;
        end;
        if g = τ then begin
            if touched(Aσ⁻¹) = false then begin
                touched(Aσ⁻¹) := true;
                gen( Aτ, σ );
                touched(Aσ⁻¹) := false;
            end;
        end else begin
            if touched(Aτ) = false then begin
                touched(Aτ) := true;
                gen( Aτ, σ );
                touched(Aτ) := false;
            end;
        end;
    end;
end {of gen};
```

Figure 6: Second refinement.

References

[CoWi] R.C. Compton and S.G. Williamson, "Doubly Adjacent Gray Codes for the Symmetric Group", TR-CS91-220, U.C. San Diego, 1991.

[HoSt] W. Holsztyński and R.F.E. Strube, "Paths and Circuits in Finite Groups", *Discrete Mathematics*, 22 (1978) 263-272.

[JiRu] M. Jiang and F. Ruskey, "Determining the Hamilton-Connectedness of Certain Vertex-Transitive Graphs", *Discrete Mathematics*, to appear.

[RuSa] F. Ruskey and C. Savage, "Hamilton Cycles which Extend Transposition Matchings in Cayley Graphs of S_n", *SIAM J. Discrete Mathematics*, to appear.

[NiWi75] A. Nijenhuis and H.S. Wilf, *Combinatorial Algorithms*, 1st ed., Academic Press, 1975.

[NiWi78] A. Nijenhuis and H.S. Wilf, *Combinatorial Algorithms*, 2nd ed., Academic Press, 1978.

[Wilf] H.S. Wilf, *Combinatorial Algorithms: An Update*, SIAM, CBMS 55, 1989.

[Witt] D.S. Witte, "On Hamiltonian Circuits in Cayley Diagrams", *Discrete Mathematics*, 38 (1982) 99-108.

[WiGa] D. Witte and A. Gallian, "A Survey: Hamiltonian Cycles in Cayley Graphs" *Discrete Mathematics*, 51 (1984) 293-304.

John Milnor's example (as presented in [WiGa]). Let n divide r^2-r+1 and define the group

$$G \;=\; \langle a, b \;\mid\; a^2 = b^3 = c^n = id,\; cb = bc^{r-1}\rangle,$$

where $c = a^{-1}b^{-1}ab$.

John Milnor's example (as presented in [HoSt]). Let n divide r^2-r+1 and define the group

$$G \;=\; \langle a, b \;\mid\; a^2 = b^3 = (ab)^6 = (abab^2)^n = id\rangle.$$

If $n = 2$ then one obtains the following digraph (isomorphic to $C_2 \times A_4$).