Searching for Symmetric Venn Diagrams

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There has been a renewed interest in Venn diagrams in recent years [1][2], and surveys have been written by Ruskey [5] and Hamburger [4]. In this short abstract we outline our efforts to enumerate all symmetric Venn diagrams on 5 and 7 curves, and to find a symmetric diagram on 11 curves. Symmetric diagrams have surprising ties to topics such as the middle-two levels problem, single-track Gray codes, and revolving-door algorithms, but perhaps their most enticing feature is the undeniable aesthetic appeal of their drawings.

An n-*Venn diagram* is a collection of n finitely intersecting simple closed curves in the plane, the removal of which divides the plane into 2^n open connected regions. Further, each region must contain a unique intersection of n interiors or exteriors of the curves. A diagram in which no more than two curves intersect in a point is said to be *simple*. In a simple Venn diagram the number of vertices is $2^n - 2$. In a *symmetric* diagram all curves are congruent and obtained by rotation by $2i\pi/n$ about a common point for $0 \leq i < n$. A necessary condition for symmetric diagrams is that n be prime (since in a symmetric diagram $n \mid \binom{n}{k}$ for all $0 < k < n$). A diagram is *polar symmetric* if it is the same when viewed through either end of the axis in its canonic stereographic projection.

The *weight* of a region is the number of curves that contain it. With each diagram V we associate the *directed dual graph* $\vec{D}(V)$, obtained by orienting each edge of the dual graph from the region of greater weight to the region of smaller weight. A Venn diagram is *monotone* if its directed dual graph has a unique source and a unique sink. Monotone diagrams are precisely those that are isotopic to diagrams whose curves are all convex.

We have developed a number of techniques and data structures for searching for Venn diagrams. Our techniques have led to an exhaustive enumeration of all symmetric Venn diagrams for $n = 5$ and enumeration of various classes of diagrams for $n = 7$. Grünbaum's problem of finding a symmetric diagram for $n = 11$ remains as elusive as ever.

The key to all these constructions is a pie slice of angle $2\pi/n$ taken from the diagram. Any slice induces a permutation σ of the curves. In order for the curves to return to their original position after a complete rotation it is necessay that σ^n is the identity. Since n is prime, this implies that σ must be a circular permutation. We now discuss our four techniques.

For monotone simple diagrams, generate combinations. Encode the diagram as a collection of $n-1$ combinations, subject to some other side constraints; for example, that the number of intersections represented is $(2^n - 2)/n$. Generate these combinations, checking first whether the permutation induced is circular and, if so, whether the collection is a Venn diagram. This technique was used to show that there are 23 simple monotone Venn diagrams, 6 of which are polar symmetric (5 found by Edwards [2]).

Pick and then color a curve pattern. This works particularly well for non-simple monotone diagrams (and by hand). Pick a pattern of uncolored curves that has the proper number of regions and then try to color the curves so that they form a Venn diagram. The key idea is to encode each region so that all regions are distinct under the group action of the cyclic group on the curve colors. This technique was used to find some minimum vertex 5-Venn diagrams and and some of the first non-simple 7-Venn diagrams.

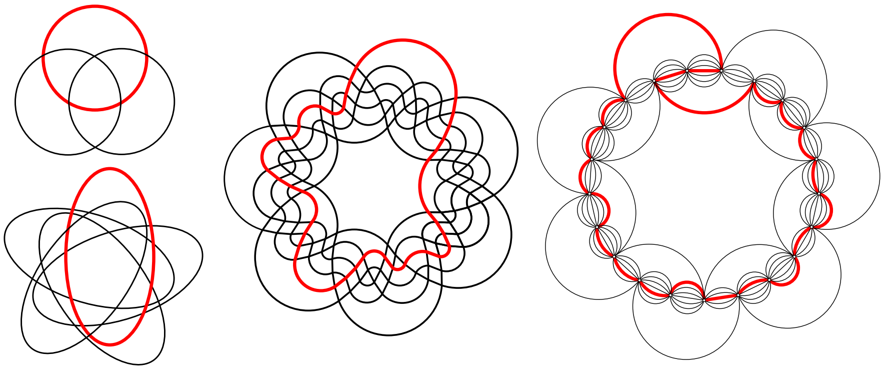

Figure 1: Symmetric Venn diagrams for $n = 3, 5, 7$. The first three are polar symmetric, the last is non-simple, all are monotone.

Permute and then stretch randomly. This technique works well for simple, non-monotone diagrams. Pick a circular permutation and place segments that induce the permutation. The number of resulting intersections is equal to the number of inversions of the permutation. Now apply an operation to increase the number of intersections to the desired amount ($(2^n - 2)/n$ for simple diagrams). Thereafter apply random (or hill-climbing) moves that preserve the number of intersections. This technique was used to find 33 simple non-monotone 7-Venn diagrams.

Weave across the pie. Successive curves are woven from left to right across the pie slice. This technique was used to show that there are 243 symmetric 5-Venn diagrams, 1 of which is simple, and 31 of which are rigid (vertices cannot be separated). It has also been to find many symmetric 7-Venn diagrams; for example, more than 100 minimum vertex diagrams (21 vertices), including 1 that is rigid. The data structure used in the last two techniques is a modification of the quad-edge data structure of Guibas and Stofli [3].

Venn diagrams present a challenge for graph-drawers. Spring algorithms tend to produce drawings that have crossing edges and/or are not symmetric. The Tutte embedding [6] produces reasonable drawings of simple diagrams but its requirement of 3-connectedness makes it unsuitable for non-simple diagrams.

References

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