A Gray Code for Combinations of a Multiset

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Abstract

Let $\mathbf{C}(k; n_0, n_1, \ldots, n_t)$ denote the set of all k-element combinations of the multiset consisting of n_i occurrences of i for $i = 0, 1, \ldots, t$. Each combination is itself a multiset. For example, $\mathbf{C}(2; 2, 1, 1) = \{\{0, 0\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\}$. We show that multiset combinations can be listed so that successive combinations differ by one element. Multiset combinations simultaneously generalize compositions and combinations, for which minimal change listings, called Gray codes, are known.

1 Introduction

By $\mathbf{C}(k; n_0, n_1, \ldots, n_t)$ we denote the set of all ordered $(t+1)$-tuples $(x_0, x_1, \ldots, x_t)$ satisfying

$$x_0 + x_1 + \cdots + x_t = k \quad \text{where} \quad 0 \leq x_i \leq n_i \ \text{ for } i = 0, 1, \ldots, t. \tag{1}$$

We assume throughout the paper that n_i is a positive integer for $i = 0, 1, \ldots, t$. Solutions to (1) are referred to as *combinations of a multiset* because they can be regarded as k-subsets of the multiset consisting of n_i copies of i for $0 \leq i \leq t$. Solution $(x_0, x_1, \ldots, x_t)$ corresponds to the k-subset consisting of x_i copies of i for $0 \leq i \leq t$. For example, $\mathbf{C}(2; 2, 1, 1)$ corresponds to $\{\{0, 0\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\}$. When each n_i is 1, the set $\mathbf{C}(k; n_0, n_1, \ldots, n_t)$ is simply the set of all combinations of $t + 1$ elements chosen k at a time.

Alternatively, the elements of $\mathbf{C}(k; n_0, n_1, \ldots, n_t)$ can be regarded as placements of k identical balls into $t + 1$ labeled boxes where the ith box can hold at most n_i balls, i.e., compositions of k into $t+1$ parts in which the ith part is at most n_i. Under this interpretation, $\mathbf{C}(2; 2, 1, 1) = \{(0\ \ 1\ \ 1),\ (1\ \ 0\ \ 1),\ (1\ \ 1\ \ 0),\ (2\ \ 0\ \ 0)\}$. When each n_i is at least k, $\mathbf{C}(k; n_0, n_1, \ldots, n_t)$ is the set of all compositions of k into $t + 1$ parts.

These are frequently occurring combinatorial objects, and it is natural to consider efficient algorithms for generating them. In particular, we will focus on listing multiset combinations in a minimal change order. Minimal change listings of combinatorial objects are called *Gray codes*, after Frank Gray, who patented a scheme for listing n-bit strings so that successive strings differ in just one bit [9].

We adopt the convention that lower case bold letters represent $(t+1)$-tuples; e.g., $\mathbf{a} = (a_0, a_1, \ldots, a_t)$. Two elements $\mathbf{x}$, $\mathbf{y}$ of $\mathbf{C}(k; \mathbf{n})$ are *adjacent* if there are indices p and q such that $x_i = y_i$ for $i \neq p, q$, where in addition $x_p = y_p + 1$ and $x_q = y_p - 1$. This concept of adjacency seems to be the most natural one for solutions in integers to an equation of the form $x_0 + x_1 + \cdots + x_t = k$, possibly subject to some other side constraints. It has been applied to combinations (e.g., Bitner, Ehrlich and Reingold [1] or Eades and McKay [4]), compositions (e.g., as attributed to Knuth in Wilf [14]), and to integer partitions (e.g., Savage [13], Rasmussen, Savage and West [11]). For each of these classes, it was shown that there is an exhaustive listing of the elements in which successive elements on the list are adjacent under this criterion.

In this paper, we seek a listing of the elements of $\mathbf{C}(k;\mathbf{n})$ in which successive elements are adjacent. Such a listing will be referred to as a *Gray code for combinations of a multiset.*

The problem of efficiently generating the combinations of a multiset (in any order) was also given in Reingold, Nievergelt, and Deo [12] (pg. 201, exercise 23), but the associated solutions manual by Fill and Reingold [8] contains no solution. A loopfree algorithm, called COMPOMAX, was developed by Ehrlich [5], [6]. In Ehrlich's algorithm successive solutions differ in two adjacent positions, but in those positions the elements may differ by more than one from their previous values. Roelants van Baronaigien, Ruskey, and Engels [7] provide a simpler constant amortized time generation algorithm which, however, is not a Gray code; it generates the combinations in lexicographic order.

2 The Construction

Define a multiset combination $\mathbf{x}$ to be *left extreme* if $x_i > 0$ implies either $i = 0$ or $x_{i-1} = n_{i-1}$ and *right extreme* if $x_i > 0$ implies $i = n$ or $x_{i+1} = n_{i+1}$. Equivalently, $\mathbf{x}$ is *left (right) extreme* if it is maximum (minimum) in $\mathbf{C}(k;\mathbf{n})$ in the lexicographic ordering of $(t+1)$-tuples. For example, in $\mathbf{C}(2;2,1,1)$, $(2,\ 0,\ 0)$ is left extreme and $(0,\ 1,\ 1)$ is right extreme. Let $n = n_0 + n_1 + \cdots + n_t$. Left and right extreme multiset combinations are unique, although a multiset combination is both left and right extreme if $k = 0$ or $k = n$, since in these cases $\mathbf{C}(k;\mathbf{n})$ has only one element. We call a Gray code for combinations of a multiset *extreme* if it starts with the right extreme combination and ends with the left extreme combination.

Lemma 1 *If both $\mathbf{C}(k;\mathbf{n})$ and $\mathbf{C}(k+1;\mathbf{n})$ are nonempty then their right (left) extreme elements differ in only one position, and in that position by one.*

Proof. If $(x_0, x_1 \ldots, x_t)$ is right extreme for $\mathbf{C}(k;\mathbf{n})$, there is an index j, $0 \le j \le t$ such that $x_i = 0$ for $0 \le i < j$ and $x_i = n_i$ for $t \ge i > j$. Then the right extreme element of $\mathbf{C}(k+1;\mathbf{n})$ is obtained by adding 1 to x_j if $x_j < n_j$ and otherwise by adding 1 to x_{j-1}. The left extreme case is similar. $\square$

Our Gray code construction is recursive, with the basis cases covered in Lemmas 2 and 3 and the general case in Theorem 1. If $\mathbf{x} \in \mathbf{C}(k;\mathbf{n})$, then x_t must satisfy

$$\max(0, k - n + n_t) \le x_t \le \min(n_t, k).$$

Thus, when $t \geq 1$, $\mathbf{C}(k;\mathbf{n})$ can be partitioned as the disjoint union

$$\mathbf{C}(k;\mathbf{n}) \;=\; \biguplus_{x_t=\max(0,k-n+n_t)}^{\min(n_t,k)} \mathbf{C}(k-x_t;n_0,\ldots,n_{t-1}).$$

Let $G(k;\mathbf{n})$ be the graph whose vertex set is $\mathbf{C}(k;\mathbf{n})$, and where edges exist between vertices that are adjacent. A Gray code for $\mathbf{C}(k;\mathbf{n})$ is a Hamilton path in $G(k;\mathbf{n})$.

Lemma 2 *There is an extreme Gray code listing of* $\mathbf{C}(k;\mathbf{n})$ *when* $t = 1$ *and* $0 \leq k \leq n$.

Proof: Let $p = \max(0, k - n_1)$ and $q = \max(0, k - n_0)$. In this case $G(k; n_0, n_1)$ is a path and there is only one possible list as shown below.

$$(p, k-p), (p+1, k-p-1), \ldots, (k-q, q).$$

$\square$

Lemma 3 *There is a Gray code listing of* $\mathbf{C}(k;\mathbf{n})$ *when* $t = 2$*, which is extreme unless* $n_0 = n_2 = 1$ *and* $2 \leq k \leq n_1$.

Proof: Here it is helpful to consider the graph $G(k; n_0, n_1, n_2)$, one instance of which is shown in Figure 2. The solid squares represent the extreme vertices. Some variation of this figure can occur, depending upon the location of the lines $x_0 + x_1 = k$ and $x_0 + x_1 = k - n_2$.

Figure 2 shows a construction that yields a Hamilton path between the two extreme vertices. The path is constructed a column at a time, from left-to-right, with the possible exception of the rightmost two columns, where a zig-zag path from bottom-to-top may be necessary. This construction works so long as $n_0 > 1$. If $n_0 = 1$, then it is possible to zig-zag from one extreme to the next, unless $n_2 = 1$ and $n_1 \geq k \geq 2$. In this case, $\mathbf{C}(k; 1, n_1, 1)$ has four elements which can be listed in Gray code order as $(0, k-1, 1)$, $(0, k, 0)$, $(1, k-1, 0)$, $(1, k-2, 1)$. $\square$

Lemma 4 *If there is an extreme Gray code for* $\mathbf{C}(k;\mathbf{n})$*, then there is an extreme Gray code for* $\mathbf{C}(n-k;\mathbf{n})$.

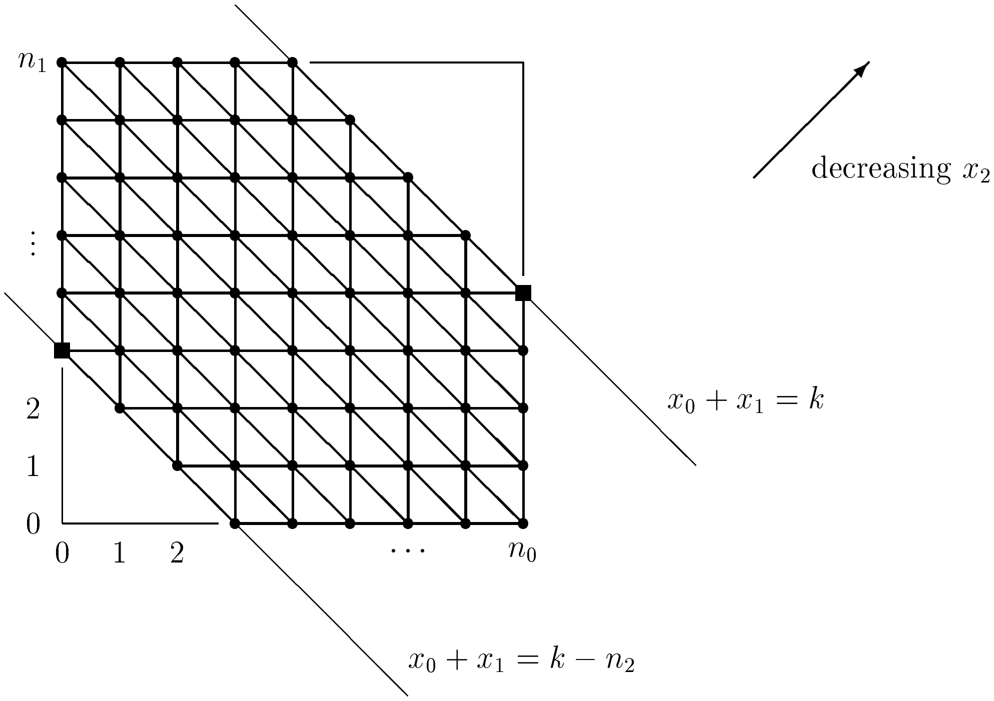

Figure 1: The graph $G(k; n_0, n_1, n_2)$.

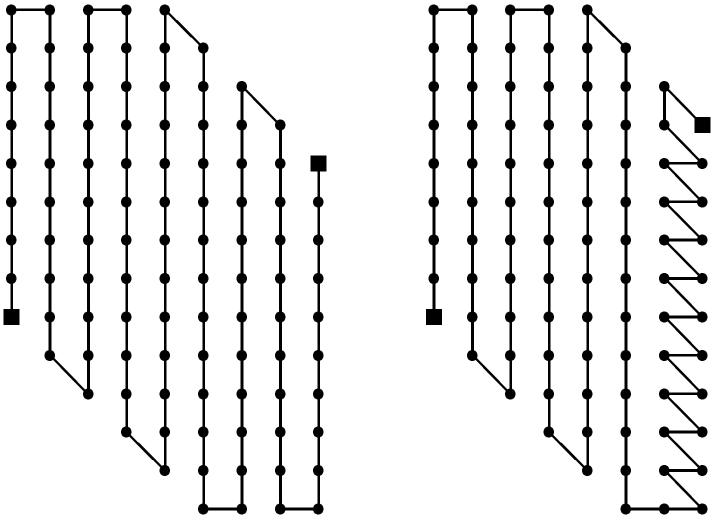

Figure 2: Extreme Gray codes (a) n_0 even, (b) n_0 odd.

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Figure 3: Path matices arising in the proof of Theorem 1.

Proof: Given $\mathbf{x} = (x_0, x_1, \ldots, x_t)$ in $\mathbf{C}(k; \mathbf{n})$, define $\phi(\mathbf{x}) = \mathbf{y}$, where $y_i = n_i - x_i$ for $i = 0, 1, \ldots, t$. Now observe that ϕ is a graph isomorphism between $G(k; \mathbf{n})$ and $G(n - k; \mathbf{n})$. It is also clear that a left extreme vertex in $G(k; \mathbf{n})$ is right extreme in $G(k; \mathbf{n})$, and vice-versa. $\square$

The following technical lemma will be used in Theorem 1 below to show the existence of a Gray code in $\mathbf{C}(k; \mathbf{n})$. Let $\big[1\big]$ denote the 4 by 4 Boolean matrix with all entries 1.

Lemma 5 *Let $A, B, C,$ and D be the four Boolean matrices in Figure 3. Let M be the Boolean product, $M = M_1 M_2 \cdots M_k$, where $k \geq 2$ and $M_i \in \{A, B, C, D, \big[1\big]\}$ for $1 \leq i \leq k$. Then $M = \big[1\big]$ unless $M_1 = M_2 = \cdots = M_k$.*

Proof. It can be checked that the product of any two distict matrices from $\{A, B, C, D, \big[1\big]\}$ is $\big[1\big]$. Thus, $M = \big[1\big]$ if at least two different matrices are factors in the product $M_1 = M_2 = \cdots = M_k$. $\square$

Note that in constructing a Gray code for $\mathbf{C}(k; \mathbf{n})$, it suffices to consider the cases where $\mathbf{n}$ is nondecreasing : $n_0 \leq n_1 \leq \ldots \leq n_t$. For, if π is a permutation of $\{0, \ldots, t\}$ and $\mathbf{n}_\pi = (n_{\pi(0)}, n_{\pi(1)}, \ldots, n_{\pi(t)})$, a Gray code for $\mathbf{C}(k; \mathbf{n}_\pi)$ can be obtained from a Gray code for $\mathbf{C}(k; \mathbf{n})$ by permuting the coordinates by π. However, this permutation does not preserve extreme elements. Theorem 1 below guarantees an extreme Gray code for all $\mathbf{C}(k; \mathbf{n})$ with nondecreasing $\mathbf{n}$, and therefore a Gray code (not necessarily extreme) for all multiset combinations.

Theorem 1 *For all $k \geq 0$ and non-decreasing $\mathbf{n}$, there is an extreme Gray code for $\mathbf{C}(k; \mathbf{n})$.*

$\mathcal{L}$: (Extreme Gray code for $\mathbf{C}(5; 2+2, 2+4) = \mathbf{C}(5; 4, 6)$)				
i	(w_i, z_i)	$C_i = \mathbf{C}(z_i; 2, 2) \times \mathbf{C}(w_i; 2, 4)$	$\mathcal{M}_z^{(i)}$ Extreme Gray code for $\mathbf{C}(z_i; 2, 2)$	$\mathcal{M}_w^{(i)}$ Extreme Gray code for $\mathbf{C}(w_i; 2, 4)$
1	(0,5)	$\mathbf{C}(0; 2, 2) \times \mathbf{C}(5; 2, 4)$	(0,0)	(1,4) (2,3)
2	(1,4)	$\mathbf{C}(1; 2, 2) \times \mathbf{C}(4; 2, 4)$	(0,1) (1,0)	(0,4) (1,3) (2,2)
3	(2,3)	$\mathbf{C}(2; 2, 2) \times \mathbf{C}(3; 2, 4)$	(0,2) (1,1) (2,0)	(0,3) (1,2) (2,1)
4	(3,2)	$\mathbf{C}(3; 2, 2) \times \mathbf{C}(2; 2, 4)$	(1,2) (2,1)	(0,2) (1,1) (2,0)
5	(4,1)	$\mathbf{C}(4; 2, 2) \times \mathbf{C}(1; 2, 4)$	(2,2)	(0,1) (1,0)

Figure 4: Example: Gray code construction in Theorem 1 for $\mathbf{C}(5; 2, 2, 2, 4)$.

Proof: The case $t = 0$ is trivial or empty, and the cases $t = 1$ and $t = 2$ are covered by Lemmas 2 and 3. So we assume that $t \geq 3$ and proceed by induction on t. By Lemma 4 we may assume also that $k \leq n/2$. Let $\mathbf{x}$ be right extreme. Since $\mathbf{x}$ has at least four coordinates, arranged in nondecreasing order, $x_0 = x_1 = 0$. Now consider the problem of listing the solutions to $z + w = k$, where $0 \leq z \leq n_0 + n_1$ and $0 \leq w \leq n_2 + \cdots + n_t$. By Lemma 2, there is an extreme Gray code for this set, namely

$$\mathcal{L} = (z_1, w_1), (z_2, w_2), \ldots, (z_m, w_m),$$

where $(z_i, w_i) = (i - 1, k - i + 1)$ and $w_m = \max(0,\ k - (n_0 + n_1))$. For $1 \leq i \leq m$, let C_i be the subset of $\mathbf{C}(k; \mathbf{n})$ defined by

$$C_i = \mathbf{C}(z_i; n_0, n_1) \times \mathbf{C}(w_i; n_2, n_3, \ldots n_t),$$

and let G_i be the subgraph of $\mathbf{G}(k; \mathbf{n})$ induced by C_i. Then

$$\mathbf{C}(k; \mathbf{n}) = \bigcup_{i=1}^{m} C_i.$$

(See Figure 4 for an example.)

To prove the existence of a Gray code for $\mathbf{C}(k; \mathbf{n})$, we show that each C_i has a Gray code, $\mathcal{L}_i$, satisfying:

(i) For $1 \leq i < m$, the last element of $\mathcal{L}_i$ is adjacent to the first element of $\mathcal{L}_{i+1}$,

(ii) the first element of $\mathcal{L}_1$ is right extreme, and

(iii) the last element of $\mathcal{L}_m$ is left extreme.

Then $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m$ is an extreme Gray code for $\mathbf{C}(k; \mathbf{n})$.

By induction, there exist extreme Gray codes

$$\mathcal{M}_z^{(i)} \text{ for } \mathbf{C}(z_i; n_0, n_1) \text{ from } r_z^{(i)} \text{ to } l_z^{(i)} \quad \text{and} \tag{2}$$

$$\mathcal{M}_w^{(i)} \text{ for } \mathbf{C}(w_i; n_2, \ldots, n_t) \text{ from } r_w^{(i)} \text{ to } l_w^{(i)}. \tag{3}$$

(See example in Figure 4.) Thus, G_i has as a spanning subgraph a grid graph, denoted H_i, in which $(x_0, \ldots x_t)$ is adjacent to $(y_0, \ldots y_t)$ if either

$$(x_0, x_1) \text{ is the successor of } (y_0, y_1) \text{ on } \mathcal{M}_z^{(i)} \text{ and } x_j = y_j \text{ for } i > 1, \quad \text{or}$$

$$(x_2, \ldots x_t) \text{ is the successor of } (y_2, \ldots y_t) \text{ on } \mathcal{M}_w^{(i)} \text{ and } x_i = y_i \text{ for } i = 0, 1.$$

(See Figure 5 for the grids in the example of Figure 4). The four corners of H_i correspond to the four different pairs of extreme elements: $(r_z^{(i)}, r_w^{(i)})$, $(r_z^{(i)}, l_w^{(i)})$, $(l_z^{(i)}, l_w^{(i)})$, and $(l_z^{(i)}, r_w^{(i)})$; we refer to those corners as rr, rl, ll, lr, respectively. (If (2) has only one element, then $rr = lr$ and $rl = ll$; if (3) has only one element, then $rr = rl$ and $lr = ll$.) Note that by Lemma 1, for $1 \leq i < m$, corresponding corners of H_i and H_{i+1} are adjacent.

Our strategy is to connect hamilton paths through successive grid graphs at their corners in such a way that we end and start at the extreme elements of $\mathbf{C}(k; \mathbf{n})$.

Depending upon the dimensions of a grid graph, H_i, only certain corners of H_i may be connected by Hamilton paths in H_i. The four possibilities are summarized by the path matrices shown in Figure 3. below. Both the rows and columns of these matrices are indexed rr, rl, ll, lr. A '1' entry indicates that there is a Hamilton path between the corners, and a 0 indicates that there is no such Hamilton path. Matrix A corresponds to an r by c grid graph with r and c both odd; matrix B to r odd, c even; matrix C to r even, c odd; matrix D to r even, c even. If $r = c = 1$, matrix $\mathbf{1}$ represents the possible hamilton paths. Otherwise, if $r = 1$, use matrix B or if $c = 1$, use matrix C.

Let P_i be the path matrix for H_i. For $1 \leq i \leq m$, we want a hamilton path q_i in H_i which starts and ends at a corner of H_i such that q_1 starts in corner rr (right extreme,) q_m ends in corner ll (left extreme,) and for $1 \leq i < m$, q_i ends in the corner of H_i corresponding to the corner of H_{i+1} in which q_{i+1} begins. This is possible if and only if the product $P = P_1 P_2 \cdots P_m$ has entry $P[rr, ll]$ equal to 1. By Lemma 5, it suffices to show that either $m = 1$ and $P_1[rr, ll] = 1$ or that the sequence $P_1, P_2, \ldots P_m$ contains at least two distinct elements.

The first entry of $\mathcal{L}$ is always $(z_1, w_1) = (0, k)$. Thus, $C_1 = \mathbf{C}(0; n_0, n_1) \times \mathbf{C}(k; n_2, \ldots, n_t)$. The first factor contains only the element $(0, 0)$, so H_1 has only one row and therefore $P_1 \in \{[1],\ B\}$. If $m = 1$, the result follows. Otherwise, $m \geq 2$, $(z_2, w_2) = (1, k-1)$ and $C_2 = \mathbf{C}(1; n_0, n_1) \times \mathbf{C}(k-1; n_2, \ldots, n_t)$. The first factor of C_2 contains exactly two elements $(0, 1)$ and $(1, 0)$, so that H_2 has two rows and therefore $P_2 \notin \{\ 1, B\}$. Then by Lemma 5 the product $P_1 P_2 \cdots P_m = [1]$ since P_1 and P_2 are distinct. $\square$

3 Final Remarks

An open problem is to find Gray codes where the x_i that change are *adjacent*. In that case the underlying graph is bipartite and we may compute the difference in the sizes

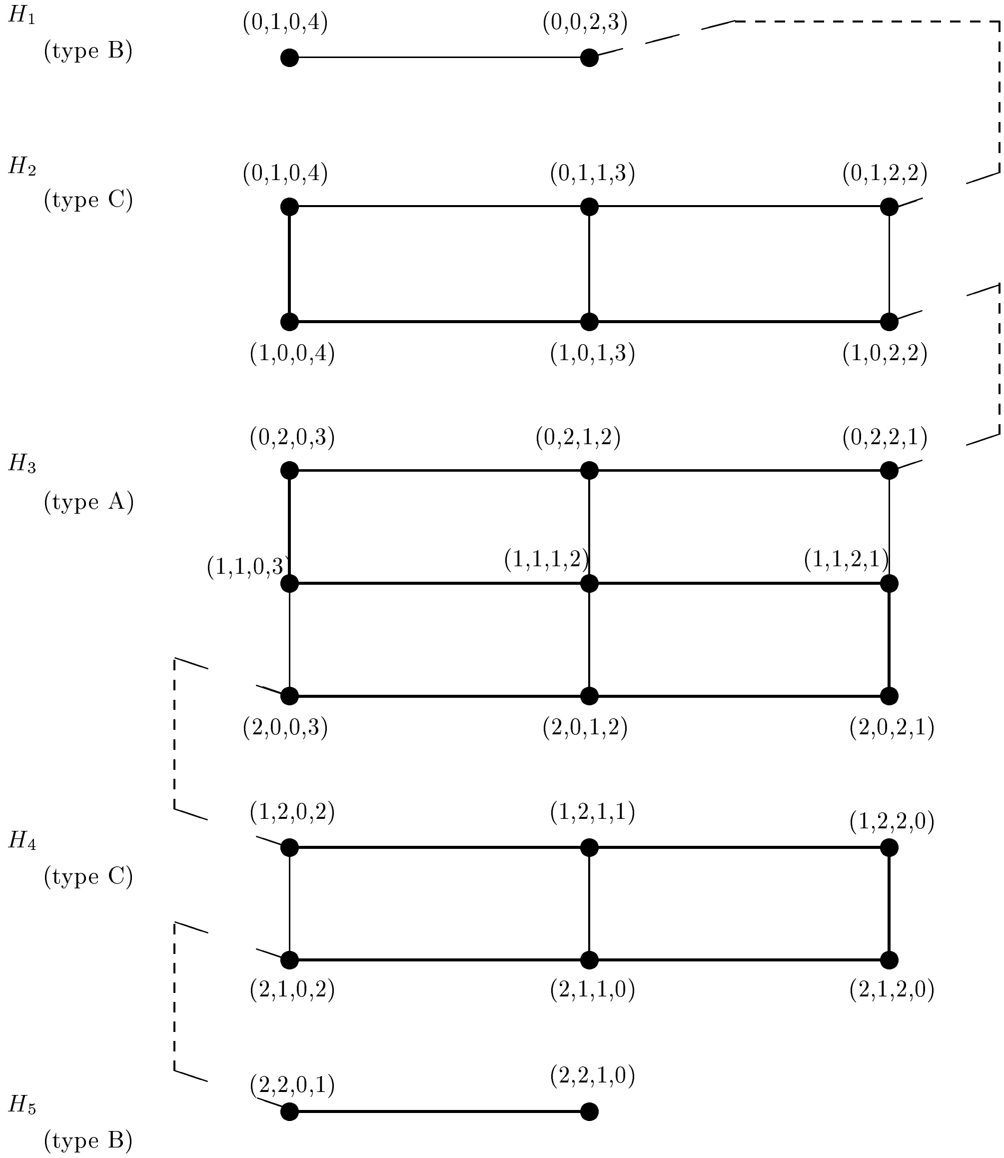

Figure 5: The grid graphs for the example in Figure 4. Gray code in $\mathbf{C}(5; 2, 2, 2, 4)$ can be obtained by using indicated adjacency links between grids.

of the partite sets. If the difference is greater than one, then there is no Gray code.

Given a solution $\mathbf{x}$ to (1), define the *weight*, $w(\mathbf{x})$, of $\mathbf{x}$ as

$$w(x_0, \ldots, x_t) = x_0 + x_2 + \cdots = \sum_{i \text{ even}} x_i.$$

Observe that moving a ball from one box to an adjacent box changes the parity of the weight. Define the *parity difference* $d(k; \mathbf{n})$ to be

$$d(k; \mathbf{n}) = \sum_{\mathbf{x} \in \mathbf{C}(k; \mathbf{n})} (-1)^{w(\mathbf{x})}$$

The parity difference satisfies the following recurrence relation.

$$d(k; \mathbf{n}) = \begin{cases} +1 & \text{if } t = 0 \text{ and } k \text{ even} \\ -1 & \text{if } t = 0 \text{ and } k \text{ odd} \\ \sum_{i=\max(0,k-n+n_t)}^{\min(n_t,k)} d(k-i; n_0, \ldots, n_{t-1}) & \text{if } t > 0 \text{ odd} \\ \sum_{i=\max(0,k-n+n_t)}^{\min(n_t,k)} (-1)^i d(k-i; n_0, \ldots, n_{t-1}) & \text{if } t > 0 \text{ even} \end{cases}$$

This recurrence can be solved if all $n_i = 1$. See, for example, Buck and Wiedemann [2], Eades, Hickey and Read [3], Ko and Ruskey [10]. Then

$$d(k; \mathbf{1}) = \begin{cases} 0 & \text{if } t \text{ and } k \text{ are odd} \\ \binom{\lfloor (t+1)/2 \rfloor}{\lfloor k/2 \rfloor} & \text{otherwise.} \end{cases}$$

By similar reasoning, it can also be solved when all $n_i = \infty$.

$$d(k; \infty) = \begin{cases} 0 & \text{if } t \text{ and } k \text{ are odd} \\ \binom{\lfloor (t+k)/2 \rfloor}{\lfloor k/2 \rfloor} & \text{otherwise} \end{cases}$$

We close by mentioning three open problems:

- Determine exact conditions under which the parity difference is zero.
- Develop an algorithm for generating the Hamilton path in $G(k; \mathbf{n})$ that runs in time proportional to $|\mathbf{C}(k; \mathbf{n})|$.
- This paper shows the existence of a Hamilton path in $G(k; \mathbf{n})$. We believe that there is a Hamilton cycle in $G(k; \mathbf{n})$ whenever $t > 1$.

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