Convex Drawings of Intersecting Families of Simple Closed Curves

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Abstract

A FISC or *family of intersecting simple closed curves* is a collection of simple closed curves in the plane with the properties that there is some open region common to the interiors of all the curves, and that every two curves intersect in finitely many points.

Let $\mathcal{F}$ be a FISC. Intersections of the curves represent the vertices of a plane graph, $\mathrm{G}(\mathcal{F})$, whose edges are the curve arcs between vertices. The directed dual of $\mathrm{G}(\mathcal{F})$, denoted $\vec{\mathrm{D}}(\mathcal{F})$, is the dual graph of $\mathrm{G}(\mathcal{F})$, but with edges oriented to indicate inclusion in fewer interiors of the curves. A convex drawing of $\mathrm{G}(\mathcal{F})$ is one in which every curve is convex. The graph $\mathrm{G}(\mathcal{F})$ has a convex drawing if there is some FISC $\mathcal{C}$ whose curves are all convex and where $\mathcal{F}$ can be transformed into $\mathcal{C}$ by a continuous transformation of the plane. We prove that $\mathrm{G}(\mathcal{F})$ has a convex drawing if and only if $\vec{\mathrm{D}}(\mathcal{F})$ contains only one source and only one sink.

1 Results

When does a family of simple Jordan curves have the property that, by a continuous transformation of the plane, the curves become convex? The curves of Figure 1 admit no such transformation, but clearly some families can be drawn with convex curves.

We say that a collection of curves is *finitely intersecting* if every pair of curves intersect in finitely many points; at each intersection point, two curves may cross or not cross.

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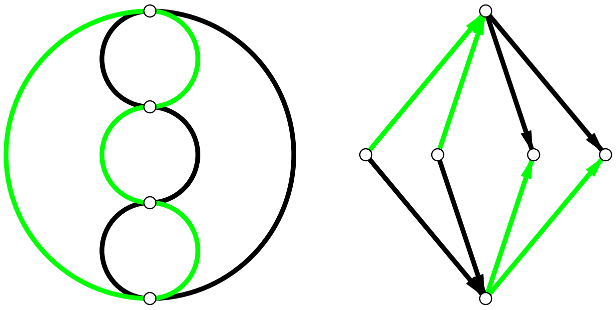

Figure 1: (a) An example of two intersecting curves that do not have a convex drawing. (b) The corresponding dual graph, which has two sources and two sinks.

Several curves may meet at the same point. A *family of intersecting simple closed curves* (or a FISC) is a collection of finitely-intersecting simple closed curves in the plane, with the property that there is some open region common to the interiors of all the curves. A FISC is *simple* if at every point of intersection exactly two curves meet, and they cross each other. A FISC is *convex* if each curve is convex.

We say that two FISCs are *isomorphic* if, by continuous transformation of the plane, one of them can be changed into the other or its mirror image [6]. If the FISC contains n curves then it is an n-FISC. Since the main result of this paper can be deduced easily for all FISCs once it is proved for those FISCs in which every curve intersects some other curve, we shall assume throughout the sequel that this condition is satisfied.

An n-FISC $\mathcal{F} = C_1, C_2, \ldots, C_n$ can be regarded as a plane graph $G = G(\mathcal{F})$, whose vertices are the intersections of the curves. The labelled edges of G are of the form $C(v, w)$, where there is a arc on curve C with intersection points v and w, and no intersection points between them on this arc. The label of the edge is i if $C = C_i$. Each face, including the outer infinite face, is called a *region* when referring to G. The *weight* of a region is the number of curves that contain it. A k-*region* is one of weight k. The unbounded 0-region is called the *outer region*. Any n-region is called an *inner region*.

The dual graph $D(\mathcal{F})$, of a FISC $\mathcal{F}$ is constructed by placing a vertex within each region of $\mathcal{F}$. For each edge of $\mathcal{F}$ that borders two regions, an edge of the dual graph is drawn which connects the vertices within these regions. Note that each of the *dual vertices* corresponds to a face in $\mathcal{F}$, and each of the *FISC vertices* corresponds to a face in $D(\mathcal{F})$. We can associate with each of the dual vertices the subset of curve interiors of the corresponding region of $\mathcal{F}$. We define the *directed* dual graph, $\vec{D}(\mathcal{F})$, by imposing a direction on each edge that indicates whether one dual vertex contains the other's subset [7]. Clearly, a directed dual graph cannot contain a directed circuit.

A FISC is *monotone* if $\vec{D}(\mathcal{F})$ has a unique source (no incoming edges) and a unique

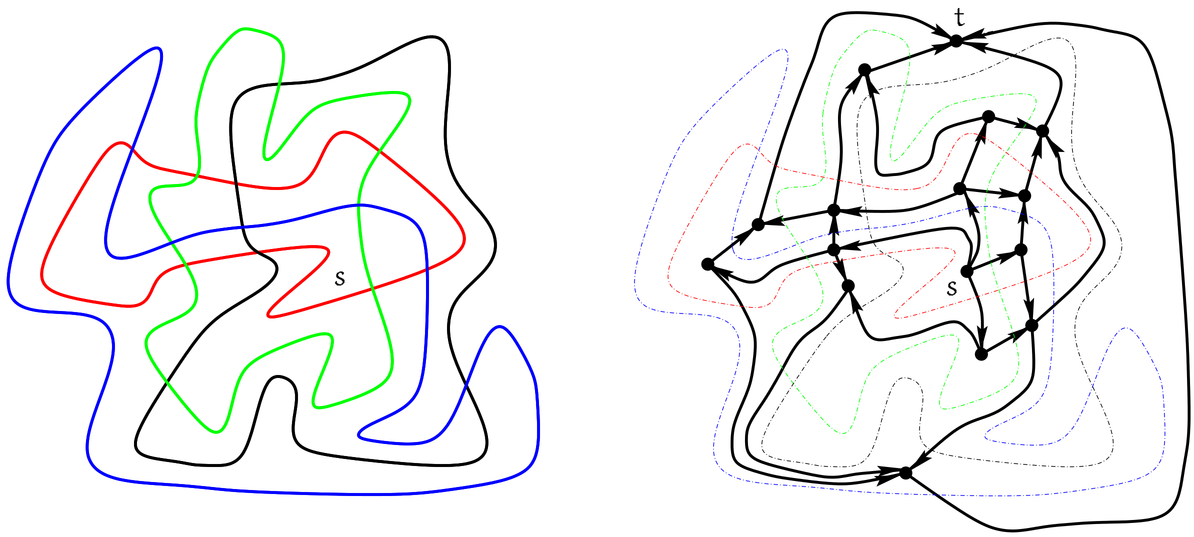

Figure 2: (a) Four curves in the plane with common region s. (b) The corresponding directed dual graph with source s and sink t.

sink (no out-going edges). The FISC shown in Figure 2(a) is monotone, as can be inferred by considering the directed dual graph shown in Figure 2(b). We will use the FISC of Figure 2 as a running example throughout this paper. It is a simple FISC.

Note that if a FISC is monotone then, for all $0 < k < n$, every k-region is adjacent to both a $(k-1)$-region and a $(k+1)$-region.

We say that two edges, both directed into the same vertex, are *separated* if, in a rotation from one incoming edge to the other, an outgoing edge is encountered, no matter in which direction the rotation is taken, clockwise or counter-clockwise.

Lemma 1.1 *Let G be a directed plane graph, with a unique source and a unique sink. If at some vertex some two incoming edges are separated, then G contains a directed circuit.*

Proof: Indeed, the two outgoing edges determine paths to the sink; these, or parts of them that contain v, yield a simple circuit composed of two outgoing arcs that start at v. In the same way, there is a simple circuit composed of two directed arcs that end incoming at v. By the Jordan curve theorem these two circuits meet at some vertex w other than v, and the directed arcs from v to w and from w to v yield the directed circuit. $\square$

Therefore, in a monotone FISC there cannot be any directed circuits.

Lemma 1.2 *If a FISC is convex, then it is monotone.*

Proof: Consider a convex n-FISC. By the definition of a FISC, it contains an n-region. This region must be connected because the intersection of convex regions is a convex region, and convex regions are clearly connected. Thus there is a unique n-region and it is a source, since every n-region in a FISC must be a source.

There is clearly at least one 0-region, namely the unique unbounded region. Let x be a point in the n-region and suppose that there was a bounded 0-region Z. Along a ray issuing from x and passing through Z let C be the last curve encountered on the boundary of Z. Since C is convex, the ray intersects C only once, and thus Z must be contained in the interior of C, a contradiction. Thus there is a unique 0-region and it is a sink.

Now consider a k-region R, where $0 < k < n$ and imagine a ray issuing from a point in the n-region and passing through R. Without loss of generality, we can position the ray so it does not intersect any of the finite number of intersection points of $\mathcal{F}$. As the ray crosses a curve just prior to entering R, it leaves a $(k+1)$-region. As the ray crosses a curve to exit R, it enters a $(k-1)$-region. Thus in the directed dual graph there is both an incoming and an outgoing edge at R. Thus any k-region, $0 < k < n$, is neither a source nor a sink.

Thus, $\mathcal{F}$ contains a unique source and a unique sink and hence is monotone. □

A FISC is *ray-monotone* if there exists a point x such that every ray emanating from x intersects each curve exactly once; in other words, each curve is star-shaped from x. The point x is called a *center* of the ray-monotone FISC.

Lemma 1.3 *If a FISC is monotone, then it is isomorphic to a ray-monotone FISC.*

Proof: Since $\vec{D}(G)$ cannot have directed circuits, it follows, by Lemma 1.1, that in the dual of a monotone FISC diagram no two incoming edges at a vertex are separated two to outgoing edges at that vertex.

It follows from this that in a monotone FISC diagram the boundary of each k-region, for $0 < k < n$, consists of an "incoming arc" (across which all directed dual edges are incoming) and an "outgoing arc" (across which all directed dual edges are outgoing). Therefore each such region can be covered by a continuous family of simple curves, one starting at each point of the incoming arc, and one ending at each point of the outgoing arc, no two of which intersect, that form a "flow" from the incoming arc to the outgoing arc. See Figure 3. These flows in the individual regions combine to disjoint continuous curves, which can be completed by arcs connecting a point in the interior of the inner region to its boundary, and by (infinite) arcs covering the outer region. Clearly, the curves can be chosen to be polygonal, and in such a way that none encounters more than one vertex.

Finally, these curves can be continuously deformed ("stretched") to straight-line rays, thus yielding an isomorphic FISC which is ray-monotone. Figure 4 shows the stretched version of our example FISC. □

Lemma 1.4 *If a FISC is ray-monotone, then it is convex.*

Proof: Let $\mathcal{F}$ be a ray-monotone n-FISC, with center x and with k crossings (points of intersection). Then $\mathcal{F}$ is isomorphic to a convex FISC $\mathcal{C}$, which can be constructed as follows.

Let L_j, for $j = 1, 2, \ldots, k$, be distinct rays issuing from x and taken in a clockwise rotation about x, which intersect each of the crossing points of $\mathcal{F}$. We may have to make small perturbations of the crossing points (or x) to achieve this condition. Let x_j denote the crossing point on L_j. On each L_j there is definite order of the n points of intersection with the curves of $\mathcal{F}$, except for the curves that pass through x_j.

We now explain how to redraw $\mathcal{F}$ so that it is convex. Let $R_1, R_2, \ldots, R_k$ be a sequence of rays, taken clockwise around a common endpoint z, and such that they divide the full angle about z into k equal parts. Ray R_j corresponds to L_j. On each ray R_j consider the segment r_j whose distance from z is between 1 and ρ, where $\rho = 1/\cos(2\pi/k)$. This value is sufficient to imply that any k-gon whose successive vertices belong to the segments r_j will be convex, regardless of the position of the vertices in the segments. See Figure 5.

Label in each r_j a set of $n - i + 1$ equidistant points, where i is the number of curves passing through the j-th intersection point x_j. Finally, connect the equally labeled points in successive R_j's by straight lines in the same order that they are encountered on successive L_j's. The final result are convex polygons that form a convex FISC $\mathcal{C}$ isomorphic to $\mathcal{F}$. The convex polygons for our example FISC are shown in Figure 6. □

Theorem 1.1 *A FISC is isomorphic to a convex FISC if and only if it is monotone.*

Proof: Immediate from Lemmas 2,3, and 4. □

2 Remarks

Our initial motivation for this paper was the determination of which Venn diagrams have convex drawings. There has been a renewed interest in Venn diagrams in the past several years (e.g., [2], [3]). Recent surveys have been written by Ruskey [7] and Hamburger [6]. The "necklace property" of Edwards' diagrams [3] is closely related to our concept of monotonicity, and our results imply that Edwards' diagrams can be drawn convexly.

An *n-independent family* in the plane is a collection of simple closed Jordan curves $\mathbf{C} = C_1, C_2, \ldots, C_n$, such that each of the 2^n sets $X_1 \cap X_2 \cap \ldots \cap X_n$ is a nonempty region [4]. Each X_i is either the bounded interior or the unbounded exterior of C_i, and this intersection can be uniquely identified by a subset of $\{1, 2, \ldots, n\}$, indicating the subset of

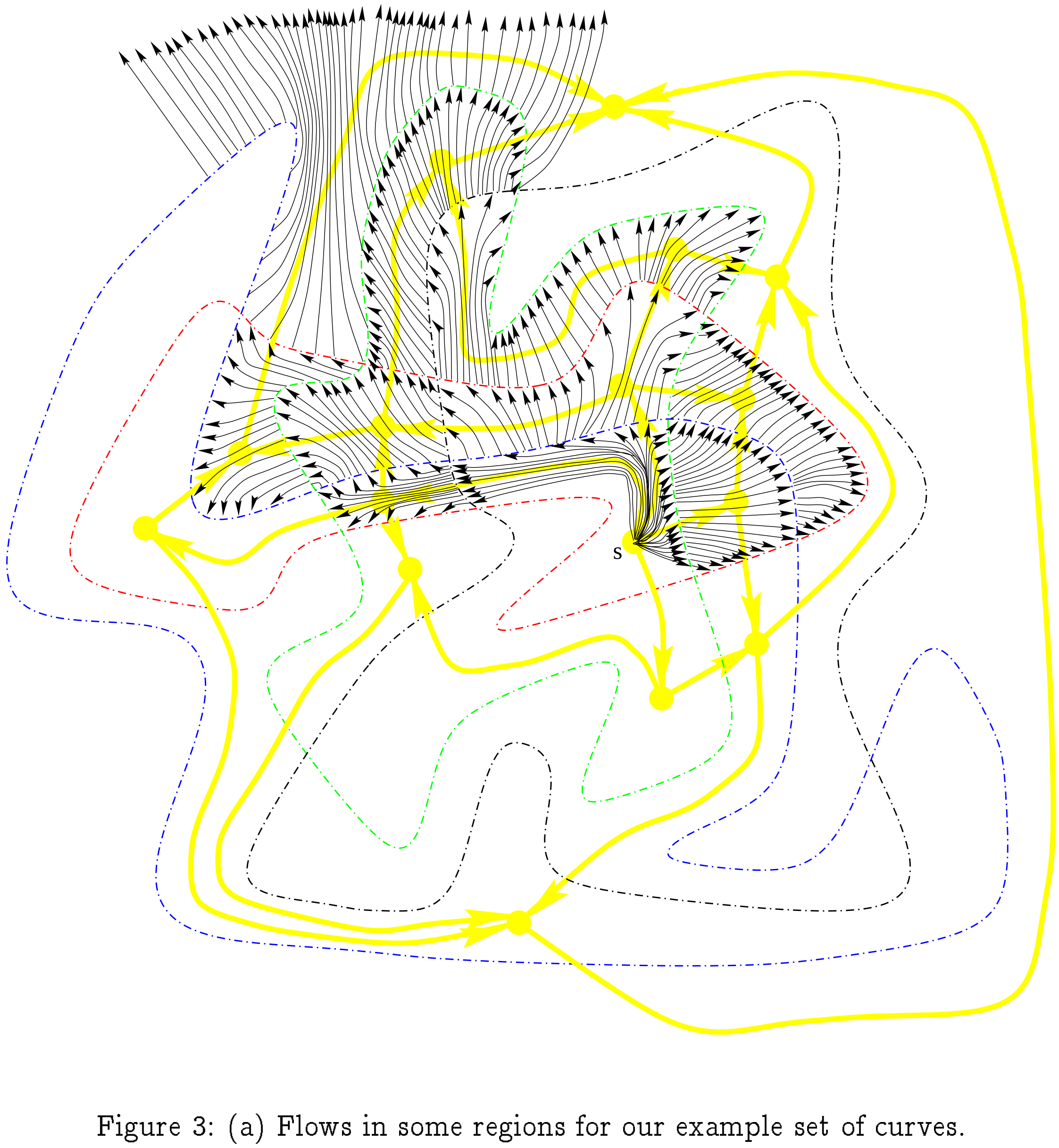

Figure 3: (a) Flows in some regions for our example set of curves.

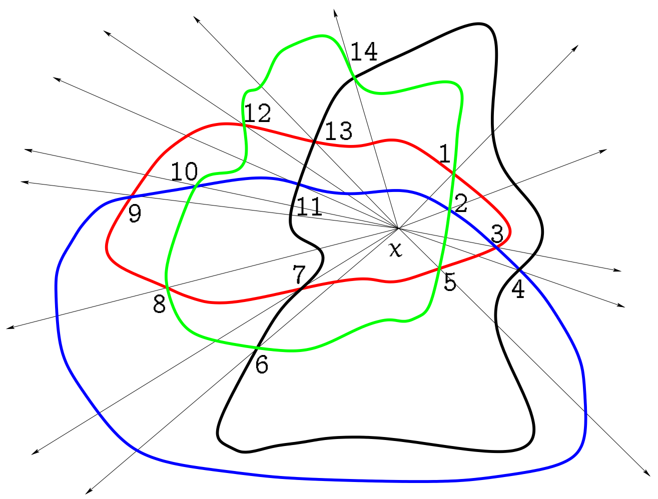

Figure 4: The curves after stretching the flows. The diagram is now ray-monotone with a center x.

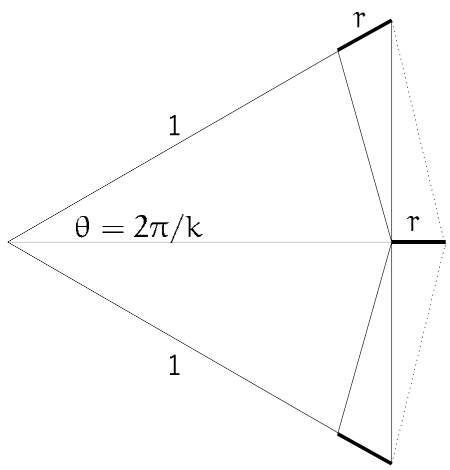

Figure 5: Note that $\rho = 1 + r = 1/\cos\theta$. All segments of the curves stay within the quadrilaterals defined by successive thickly drawn segments.

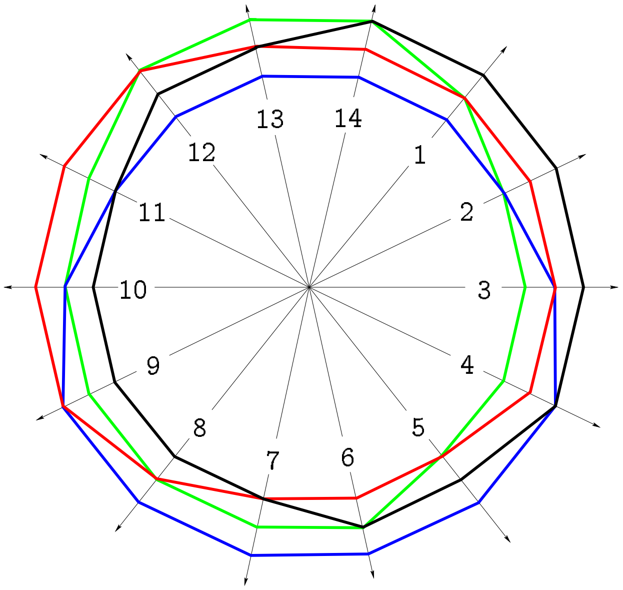

Figure 6: The example FISC with all curves drawn convex. Not drawn to scale; the value of ρ has been increased to make the curves easier to distinguish.

the indices of the curves whose interiors are included in the intersection. Pairs of curves are assumed to intersect only at a finite number of points. If all 2^n regions are connected, then an n-independent family is an n-Venn diagram. Clearly, every independent family is a FISC and thus, by our main result, every monotone independent family can be drawn with convex curves. In particular, every monotone Venn diagram can be drawn with convex curves.

Our condition that the curves intersect in only finitely many points can be relaxed. If the curves are allowed to intersect also in arcs, but only finitely many of them, then our results carry through.

We close with some open problems: (a) Characterize *all* collections of finitely intersecting simple closed curves that have convex drawings, not just those that have a region of common intersection. (b) Is the problem of determining whether a general collection of simple closed curves is isomorphic to a convex collection of simple closed curves NP-complete? (c) Given a simple monotone n-FISC with N intersections our constructions create a convex drawing consisting of nN line segments. Surely, they can be drawn with fewer segments. In the worst case, how many are required? (d) Two convex k-gons can intersect in at most $2k$ points. What is the largest value of n for which there is a convex n-Venn diagram made from k-gons? For example, it is not know whether there is a 6-Venn diagram made from triangles, or even from rectangles [5]. (e) Is there a natural generalization of our results to 3 dimensions?

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