# Completely Separating Systems of k-Sets

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**Abstract**

Dickson [On a problem concerning separating systems of a finite set, Journal of Combinatorial Theory, **7** (1969), 191–196.] introduced the notion of a completely separating set system. We study such systems with the additional constraint that each set in the system has the same size.

Let $T$ denote an $n$-set. We say that a subset $S$ of $T$ separates $i$ from $j$ if $i \in S$ and $j \notin S$. A collection of $k$-sets $\mathcal{C}$ is called a $(n, k)$-separator if, for each ordered pair $(i, j) \in T \times T$ with $i \neq j$, there is a set $S \in \mathcal{C}$ which separates $i$ from $j$. Let $R(n, k)$ denote the size of a smallest $(n, k)$-separator. For $n \geq k(k-1)$ we show that $R(n, k) = \lceil 2n/k \rceil$. We also show that $R(2^m, 2^{m-1}) \leq 2m$ and demonstrate various recursive relationships that are used to determine the exact values of $R(n, k)$ for $k \leq 5$.

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# 1 Introduction

In 1961 Rényi [6] raised the problem of finding minimum separating systems in the context of solving certain problems in information theory. Subsequently, several variants have been treated in the literature. It is the purpose of this paper to introduce and partially solve another natural variant of the problem. We begin by introducing some notation and terminology.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. A set $S \subseteq [n]$ *separates* $i$ from $j$ if $i \in S$ and $j \notin S$. A collection $\mathcal{C}$ of subsets of $[n]$ is a *separator* if, for each $i, j \in [n]$ with $i \neq j$, there is a set $S$ in $\mathcal{C}$ that separates $i$ from $j$. If, for each $(i, j) \in [n] \times [n]$ with $i \neq j$, there is a set $S$ in $\mathcal{C}$ that separates $i$ from $j$ *and* a set $T$ that separates $j$ from $i$, then we call $\mathcal{C}$ a *complete separator*. If in addition, the sets $S$ and $T$ that completely separate $i, j$ are required to be disjoint, then $\mathcal{C}$ is a *total separator*. Observe that any complete separator is a separator, but not vice-versa. E.g., $\{\{1, 2\}, \{1, 3\}\}$ is a separator but is not a complete separator since, even though 1 is separated from 2 by $\{1, 3\}$, element 2 is not separated from 1.

The generic problem is to find separators of smallest size; i.e., containing the least number of sets. Let $S(n)$ denote the size of a smallest separator of $[n]$. Rényi [6]showed that $S(n) = \lceil \lg n \rceil$.[1] Let $S(n, {}_{\leq}k)$ and $S(n, {}_{=}k)$ denote the size of a smallest separator where each set is constrained to have at most $k$ elements and exactly $k$ elements, respectively. Katona [4] showed that these two quantities are identical and established upper and lower bounds on $S(n, k)$ (hereafter we use $S(n, k)$ to denote $S(n, {}_{=}k)$). Wegener [8] simplified the proof of Katona and slightly improved the upper bound. Their results are combined in the following theorem.

**Theorem 1 (Katona-Wegener)** *If* $k \geq \lfloor n/2 \rfloor$*, then* $S(n, {}_{\leq}k) = \lceil \lg n \rceil$. *If* $k < \lfloor n/2 \rfloor$*, then*

$$\frac{n}{k} \frac{\log n}{\log en/k} \;\leq\; S(n, {}_{\leq}k) \;\leq\; \left\lceil \frac{\log n}{\log \lceil n/k \rceil} \right\rceil (\lceil n/k \rceil - 1).$$

Completely separating systems were introduced by Dickson [3]. Let $R(n)$ denote the size of a smallest completely separating system on $[n]$. He showed that $R(n) \sim \lg n$. Spencer's [7] obtained the sharper result that

$$R(n) \;=\; \min\{t : \binom{t}{\lfloor t/2 \rfloor} \geq n\}.$$

Let $R(n, {}_{\leq}k)$ and $R(n, {}_{=}k)$ denote the size of a smallest complete separator, where each set is constrained to have at most $k$ elements and exactly $k$ elements, respectively. Cai [1] shows that

$$R(n, {}_{\leq}k) = \lceil 2n/k \rceil, \quad \text{if } n > k^2/2 \geq 2.$$

[1]By $\lg n$ we mean $\log_2 n$.

Cai [2] also generalized completely separating systems in a graph theoretic setting.

Our focus here is on complete separators, with the restriction that all the sets in the collection $\mathcal{C}$ are k-subsets of $[n]$. We call such a set system an $(n,k)$-separator. The size of a smallest $(n,k)$-separator is $R(n,{=}k)$ (which hereafter is denoted $R(n,k)$). For example, $R(6,3) \leq 4$, since $\mathcal{C} = \{\{1,2,3\}, \{1,5,6\}, \{2,4,6\}, \{3,4,5\}\}$ is a $(6,3)$-separator. In fact $R(6,3) = 4$ and for $k \neq 3$, $R(6,k) = 6$. Note that any collection $\mathcal{C}$ of k-subsets of $[n]$ that is a superset of a $(n,k)$-separator is also a $(n,k)$-separator.

Where no confusion arises we denote collections of sets in an abbreviated form, omitting the braces and the commas of the contained sets. Thus the example given earlier will be written $\mathcal{C} = \{123, 156, 246, 345\}$. Elements greater than 9 are represented using the letters A, B, etc. Given a set system $\mathcal{Q}$ and an integer $p$, we use $\mathcal{Q} + p$ to denote the set system obtained by adding $p$ to each element of each $S \in \mathcal{Q}$.

It is tempting to suppose that, as in the separating case, $R(n,{\leq} k)$ equals $R(n,k)$, which would allow us to use the results of Cai. This supposition is incorrect. Consider, for example, $R(9,{\leq}4) = 5$. (A set system meeting this equality is $\{1234, 4567, 7891, 258, 369\}$.) However, it turns out that $R(9,{=}4) = 6$.

Observe that the following inequalities hold.

$$S(n) \leq S(n,{\leq}k) = S(n,k) \leq R(n,{\leq}k) \leq R(n,k)$$

Let $T(n)$ denote the size of a smallest total separator on $[n]$. Yao [9] showed that $T(3^m) = 3m$.

## 2 Results

In this section we present our results, starting with a simple symmetry observation.

**Lemma 2** *The symmetry* $R(n,k) = R(n,n-k)$ *holds for all* $1 \leq k < n$.

**Proof**: By taking complements, a $(n,k)$-separator becomes a $(n,n-k)$-separator. □

The next lemma reflects the fact that every element must occur at least twice in a $(n,k)$-separator, when $k \neq 1$. This is the "trivial lower bound".

**Lemma 3** *For all* $2 \leq k < n$,

$$R(n,k) \geq \left\lceil \frac{2n}{k} \right\rceil. \tag{1}$$

**Proof**: For $k \geq 2$ every element of $[n]$ must appear in at least two k-sets of any $(n,k)$-separator. Thus $k \cdot R(n,k) \geq 2n$ from which (1) follows. □

A trivial upper bound is

$$R(n,k) \leq n, \tag{2}$$

which follows from consideration of the set system

$$\{\{1,\dots,k\},\{2,\dots,k+1\},\dots,\{n,1,\dots,k-1\}\}.$$

The following lemma gives a slight improvement on Lemma 3 for some cases.

**Lemma 4** *For all* $2 \leq k < n$,

$$R(n,k) \geq \left\lceil \frac{n}{k} \left\lceil \frac{n-1}{n-k} \right\rceil \right\rceil. \tag{3}$$

**Proof**: For $k \geq 2$, every element $a$ of $[n]$, in a $(n,k)$-separator, must occur without each of the other $n-1$ elements of $[n]$. Each time $a$ occurs in a k-subset, it is separated from $n-k$ elements of $[n]$. Therefore $a$ must occur in at least $\left\lceil \frac{n-1}{n-k} \right\rceil$ sets. This value replaces the constant 2 in Lemma 3. □

Note that (3) is also trivially valid for $k = 1$ (see Lemma 5). To use Lemma 4 for $k' < n/2$, we use Lemma 2 and then Lemma 4 for $k = n - k'$. This provides a better bound than Lemma 3 only for those k near $n/2$. For example, if $n = 16, k' = 7$ then $R(n,k') \geq \lceil 2n/k \rceil = 5$, by Lemma 3. For $n = 16, k = 9$, Lemma 4 gives $R(n,k) \geq 6$, so $R(16,7) \geq 6$, by Lemma 2.

Interestingly, this lemma provides an improved bound only in the case where the bound in Lemma 3 is 5. Lemma 4 improves this to 6 in most, but not all, of these cases.

**Lemma 5** *For* $n \geq 2$, $R(n,1) = n$. *For* $n \geq 3$, $R(n,2) = n$.

**Proof**: That $R(n,1) \leq n$ follows from (2). That $R(n,1) \geq n$ follows from the observation that each element of $[n]$ must appear in at least one set.

Now consider the case where $k = 2$. That $R(n,2) \leq n$ follows from (2). For the lower bound use Lemma 3. □

Note that Lemma 2 immediately gives us $R(n, n-1) = n$, for $n \geq 2$ and $R(n, n-2) = n$, for $n \geq 3$.

Use $\mathcal{C}[x]$ to denote $\{S \in \mathcal{C} : x \in S\}$. Observe that a set system $\mathcal{C}$ of k-subsets of $[n]$ is a $(n,k)$-separator if and only if $\bigcap\{S \in \mathcal{C}[x]\} = \{x\}$, for all $x \in [n]$. In particular if, for all $x \in [n]$, there are two sets $S, T \in \mathcal{C}$ such that $S \cap T = \{x\}$, then $\mathcal{C}$ is a $(n,k)$-separator. We will use this criterion in the proof of the next theorem.

Our main result is the following theorem, which tells us that the bound in Lemma 3 is attained for sufficiently large $n$ compared to $k$.

**Theorem 6** *If* $n \geq k(k-1)$ *and* $1 < k < n$*, then*

$$R(n,k) = \left\lceil \frac{2n}{k} \right\rceil.$$

**Proof**: Let $n = kp + r$ where $0 \leq r < k$. Our strategy is to create a $(n,k)$-separator so that $kp$ elements appear in exactly two sets, and the remaining elements appear at least twice. The cases $p \geq k$ and $p = k-1$ require slightly different constructions, and each of these cases is broken into subcases depending on whether $r = 0$, $0 < r \leq k/2$, or $r > k/2$.

(A) If $p \geq k$, then imagine a $p$ by $k$ matrix $M$ whose $j$th column consists of the integers $1 + (j-1)p, 2 + (j-1)p, \ldots, jp$, in that order. We form one set system $\mathcal{R} = \{R_1, R_2, \ldots, R_p\}$ by taking the rows of $M$, and another $\mathcal{C} = \{C_0, C_1, \ldots, C_{p-1}\}$ by successively stripping $k$ elements from the columns of $M$ from top-to-bottom and left-to-right. To be more precise, define

$$C_j = \{1 + jk, 2 + jk, \ldots, k + jk\} \quad \text{for } 0 \leq j \leq p-1 \text{ and}$$

$$R_i = \{i, i + p, \ldots, i + (k-1)p\} \quad \text{for } 1 \leq i \leq p.$$

Note that $\mathcal{C}$ is a partition of $[kp]$, as is $\mathcal{R}$. Furthermore $|R_i \cap C_j| \leq 1$ for all $i$ and $j$. Hence $\mathcal{C} \cup \mathcal{R}$ is a $(kp,k)$-separator. Thus if $r = 0$ we are done.

Now assume that $r > 0$. To get a $(n,k)$-separator we have to account for the remaining elements, call them $E$, where $E = [n] \setminus [kp]$. There are two subcases, $r \leq k/2$ and $r > k/2$. Take $D$ to be the diagonal elements of $M$. That is,

$$D = \{1, 2 + p, \ldots, k + (k-1)p\} = \{i + (i-1)p : 1 \leq i \leq k\}.$$

We now modify the sets in $\mathcal{R}$ by removing an element of $D$ (if possible) from each set $R_i$ and replacing it with an element of $E$ so that each element of $E$ gets used, as close as possible, the same number of times. To be unambiguous, let $\mathcal{R}' = \{R'_1, R'_2, \ldots, R'_p\}$ where

$$R'_i = \begin{cases} R_i \setminus \{i + (i-1)p\} \cup \{kp + 1 + ((i-1) \bmod r)\} & \text{if } 1 \leq i \leq k, \\ R_i & \text{if } k < i \leq p. \end{cases}$$

(i) If $r \leq k/2$ then $\mathcal{S} = \mathcal{R}' \cup \mathcal{C} \cup \{D\}$ is a $(n,k)$-separator of size $2p+1$. To show that $\mathcal{S}$ is a separator, we observe that, for each element $x \in [n]$, there are two sets containing $x$ whose intersection is $\{x\}$. If $x \in D$ then the two sets are $D$ and the set $C_j$ that contains $x$ (recall that $\mathcal{C}$ is a partition of $[kp]$). If $x \in [kp] \setminus D$, then use the two sets $C_j$ and $R_i$ that contain $x$, as in the $r = 0$ case. If $x \in [n] \setminus [kp]$, then the two sets are $R_{x-kp}$ and $R_{x-kp+r}$.

(ii) If $r > k/2$, then there are some elements of $E$ that have occurred only once in $\mathcal{S}$; we need to form another set containing these singletons. The elements $1 + kp, 2+$

$kp, \ldots, k-r+kp$ have all appeared twice, the elements $1+k-r+kp, 2+k-r+kp, \ldots, n$ have appeared only once. Define $X$ to be the singletons together with the first $2(k-r)$ elements of $D$. That is,

$$\begin{aligned} X &= \{1+k-r+kp, 2+k-r+kp, \ldots, n\} \\ &\cup \{1, 2+p, \ldots, r+(2(k-r)-1)p\}. \end{aligned} \tag{4}$$

Now $\mathcal{S} \cup \{X\}$ is a $(n,k)$-separator of size $2p+2$. This follows from the same arguments given in the $r \le k/2$ case, except for those elements $x \in X \setminus D$ (i.e., the singletons); those elements are dealt with by noting that $X \cap R_{x-kp} = \{x\}$.

(B) If $p = k-1$, then imagine a $k$ by $k$ matrix $N$ constructed from $M$ by adding an extra bottom row identical to the first row of $M$ but rotated left one position. We now form a set system $\mathcal{R} = \{R_2, R_3, \ldots, R_{k-1}\}$ by taking all rows of $N$ except the first and last, and another set system $\mathcal{C} = \{C_0, C_1, \ldots, C_{k-1}\}$ consisting of the columns of $N$. To be more precise, define

$$C_j = \{1+j(k-1), 2+j(k-1), \ldots, k+j(k-1)\} \quad \text{for } 0 \le j \le k-1 \text{ and}$$

$$R_i = \{i, i+k-1, \ldots, i+(k-1)(k-1)\} \quad \text{for } 1 \le i \le k-1.$$

The element $k+(k-1)^2$ of $C_{k-1}$ is taken to be 1. The set $R_1$ will be used only when $r > 0$.

Note that $C_j \cap C_{j+1}$ (index addition taken mod $k$) is $\{k+j(k-1)\}$ and that otherwise the intersection of two sets in $\mathcal{C}$ is empty. The union of all sets $C_j$ is $[n]$. The elements $k+j(k-1)$ do not occur in $\mathcal{R}$. In fact, $\mathcal{R}$ forms a partition of $[n] \setminus \{k+j(k-1) : 0 \le j \le k-1\}$. Since $|R_i \cap C_j| \le 1$ for all $i$ and $j$, the union $\mathcal{C} \cup \mathcal{R}$ is a $(pk,k)$-separator. Thus if $r = 0$ we are done.

Now assume now that $r > 0$. To obtain a $(n,k)$-separator we have to account for the remaining elements, call them $E$, where $E = [n] \setminus [kp]$. Again, we have two subcases, $r \le k/2$ and $r > k/2$.

(i) If $r \le k/2$ then set, for $0 \le j \le k-1$,

$$C'_j = C_j \setminus \{k+j(k-1)\} \cup \{1+kp+(j \bmod r)\}.$$

Now the intersection of two sets in $\mathcal{C}' = \{C'_0, C'_1, \ldots, C'_{k-1}\}$ is either empty or is an element of $E$. It is easy to verify that the set system $\mathcal{S} = \mathcal{C}' \cup \mathcal{R} \cup R_1$ is a $(n,k)$-separator of size $2k-1$, as required.

(ii) If $r > k/2$ then define $X$ as in (4). The set system $\mathcal{S} \cup \{X\}$ is a $(n,k)$-separator of size $2k$, as required. Again, we leave verification to the reader. □

**Lemma 7** *For* $k = 3$*,* $R(4,3) = 4$*,* $R(5,3) = 5$*, and* $R(n,3) = \lceil 2n/3 \rceil$ *if* $n \ge 6$*.*

**Proof**: Note that $R(4,3) = R(4,1) = 4$ and $R(5,3) = R(5,2) = 5$, using Lemmas 2 and 5. For $n \geq 6$, the result follows from Theorem 6. $\square$

**Lemma 8** *For* $k = 4$, $R(5,4) = 5$, $R(6,4) = 6$, $R(7,4) = 5$, $R(8,4) = 5$, $R(9,4) = 6$, $R(10,4) = 5$, $R(11,4) = 6$ *and* $R(n,4) = \lceil n/2 \rceil$ *if* $n \geq 12$.

**Proof**: The values for $R(5,4)$, $R(6,4)$, and $R(7,4)$ follow from Lemma 2 and our earlier results for $k = 1, 2, 3$.

That $R(8,4) \leq 5$ follows from considering the set system $\mathcal{C} = \{1234,\ 1678,\ 2578,\ 3568,\ 4567\}$. By Lemma 3, $R(8,4) \geq 4$. Assume $R(8,4) = 4$. Then each element occurs in exactly 2 sets. We can assume $S = \{1,2,3,4\}$ occurs. To separate the elements of $S$, with only one occurrence of each element, at least 4 more sets are required. Hence $R(8,4) \neq 4$ and so $R(8,4) = 5$.

By Lemma 3, $R(9,4) \geq 5$. We show that $R(9,4) \neq 5$ as follows. We may assume that the set $A = \{1,2,3,4\}$ occurs in a $(9,4)$-separator containing 5 sets. To separate 1234, not all others sets contain only one element of $A$, as we need to separate 56789 and $R(5,3) = 5$. Thus our separator contains a set other than $A$ with more than one element of $A$. Noting the comment before Lemma 3, no more than 2 elements of $A$ can occur together in another set, else these elements cannot be separated within the given constraints. Thus we may assume the set 1256 occurs. Then, to separate 1 and 2, 1 must occur in one set and 2 in another. Similarly, to separate 5 and 6, 5 must occur in one set and 6 in another. Hence, one of the pairs 15, 16, 25, 26 occurs again. Assume it is 15. To separate these, 5 must occur in another set without 1. This means that each of 1,2,5 occur in at least 3 sets; contradicting the fact that only 2 such elements can exist. Hence, $R(9,4)$ cannot be 5. To see that $R(9,4) = 6$, consider the set system $\{1234,\ 1235,\ 1467,\ 2568,\ 3789,\ 4569\}$.

That $R(10,4) = 5$ follows from the trivial lower bound and the set system $\{1234,\ 1567,\ 2589,\ 368A,\ 479A\}$.

That $R(11,4) = 6$ follows from the trivial lower bound and the set system $\{1234,\ 4567,\ 789A,\ B368,\ B259,\ 156A\}$.

For $k \geq 12$, the result follows from Theorem 6. $\square$

How small can $n$ get and still have the conclusion of Theorem 6 still hold?

**Lemma 9** *If* $n < \binom{k+1}{2}$, $1 < k < n$, *and* $k | 2n$, *then* $R(n,k) > 2n/k$.

**Proof**: We prove the contrapositive. If $R(n,k) \leq 2n/k$ then, by Lemma 3, we must have $R(n,k) = 2n/k$. Thus each element of $[n]$ occurs exactly twice in a separator. Without loss of generality we may assume that $\{1,\ldots,k\}$ is a member of a $(n,k)$-separator. To separate these elements, using each only once more, each of $1 \ldots k$

must appear in a set by themselves and so there are at least $k+1$ sets in the $(n,k)$-separator. As each element of $[n]$ occurs twice, $2n \geq k(k+1)$, as required. $\square$

The following lemmas show how we can use a $(n,k)$-separator to construct separators for larger values of $n$ or $k$.

**Lemma 10** *If* $R(n,k) \leq k+1$, $1 < k < n$, *then* $R(n+k+1, k+1) \leq k+2$.

**Proof**: Let $\mathcal{A} = \{A_1, A_2, \dots, A_{k+1}\}$ be a $(n,k)$-separator. Consider the set system $\mathcal{B} = \{B_0, B_1, \dots, B_{k+1}\}$ where $B_0 = \{n+1, n+2, \dots, n+k+1\}$ and $B_i = A_i \cup \{n+i\}$ for $i = 1, 2, \dots, k+1$. It is easy to verify that $\mathcal{B}$ is an $(n+k+1, k+1)$-separator. $\square$

**Lemma 11** *If* $R(n,k) \geq k+1$ *then* $R(n+k+1, k+1) \leq 1 + R(n,k)$.

**Proof**: The proof is similar to that used to prove Lemma 10, but we need to be more careful about which elements are being added to the sets of the $(n,k)$-separator $\mathcal{A}$ to ensure that the old elements $[n]$ are separated from the new elements $B_0$. Let $p = R(n,k)$ and assume $p \geq k+1$. Our goal is to construct a $(n+k+1, k+1)$-separator $\mathcal{C}$ of cardinality $p+1$. Let $\mathcal{A} = \{A_1, A_2, \dots, A_p\}$ be an $(n,k)$-separator. Consider the set system $\mathcal{B} = \{B_0, B_1, \dots, B_p\}$ with $B_0 = \{n+1, n+2, \dots, n+k+1\}$ and $B_i = S_i \cup \{b_i\}$ for $i = 1, 2, \dots, p$, where the exact values of $b_i$ are yet to be specified. Let $\mathcal{B}_i$ be the set subsystem $\{B_1, B_2, \dots, B_i\}$. We select successive values of $b_i$ so that for all $a \in [n]$, the condition

$$|\mathcal{B}_i[a] \cap B_0| \leq 1, \tag{5}$$

holds for $i = 1, 2, \dots, p$. Taking $i = p$, this suffices to prove $\mathcal{B}[a] = \{a\}$ for $a \in [n]$. To get $\mathcal{B}[a] = a$ for $a \in B_0$, we set $b_i = n+i$ for $i = 1, 2, \dots, k+1$ (this is where the condition $R(n,k) \geq k+1$ is used). Succeeding values of $b_i$ are selected as follows. Noting that $b_1, b_2, \dots, b_{k+1}$ satisfy (5), assume that $b_1, b_2, \dots, b_m$ have been selected to that (5) holds. Let

$$Q = \bigcup_{a \in A_{m+1}} \mathcal{B}_m[a] \cap B_0,$$

and pick $b_{m+1} \in B_0 \setminus Q$. The set $B_0 \setminus Q$ is non-empty since $|Q| \leq k$. The invariant (5) still holds since $\mathcal{B}_{m+1}[a] \cap B_0$ is either empty or is $\{b_{m+1}\}$ depending on whether $\mathcal{B}_m[a]$ is non-empty or not. $\square$

**Lemma 12** *Let* $p + q = n$. *Then* $R(n,k) \leq R(p,k) + R(q,k)$.

**Proof**: A $(p,k)$-separator $\mathcal{P}$ and a $(q,k)$-separator $\mathcal{Q}$ may be used to create a $(n,k)$-separator $\mathcal{C}$ by taking $\mathcal{C} = \mathcal{P} \cup (\mathcal{Q} + p)$. $\square$

Table 1: Known values of $R(n,k)$ for $2 \le n \le 48$ and $1 \le k \le \min(20, n-1)$.

| n \ k | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | [2] | | | | | | | | | | | | | | | | | | | |
| 3 | 3 | 3 | | | | | | | | | | | | | | | | | | |
| 4 | 4 | [4] | 4 | | | | | | | | | | | | | | | | | |
| 5 | 5 | 5 | 5 | 5 | | | | | | | | | | | | | | | | |
| 6 | 6 | 6 | [4] | 6 | 6 | | | | | | | | | | | | | | | |
| 7 | 7 | 7 | 5 | 5 | 7 | 7 | | | | | | | | | | | | | | |
| 8 | 8 | 8 | 6 | [5] | 6 | 8 | 8 | | | | | | | | | | | | | |
| 9 | 9 | 9 | 6 | 6 | 6 | 6 | 9 | 9 | | | | | | | | | | | | |
| 10 | 10 | 10 | 7 | 5 | [6] | 5 | 7 | 10 | 10 | | | | | | | | | | | |
| 11 | 11 | 11 | 8 | 6 | 6 | 6 | 6 | 8 | 11 | 11 | | | | | | | | | | |
| 12 | 12 | 12 | 8 | 6 | 6 | [5-6] | 6 | 6 | 8 | 12 | 12 | | | | | | | | | |
| 13 | 13 | 13 | 9 | 7 | 6 | 5-7 | 5-7 | 6 | 7 | 9 | 13 | 13 | | | | | | | | |
| 14 | 14 | 14 | 10 | 7 | 7 | 6-7 | [5-8] | 6-7 | 7 | 7 | 10 | 14 | 14 | | | | | | | |
| 15 | 15 | 15 | 10 | 8 | 6 | 6-7 | 5-9 | 5-9 | 6-7 | 6 | 8 | 10 | 15 | 15 | | | | | | |
| 16 | 16 | 16 | 11 | 8 | 7 | 6-7 | 6-8 | [5-7] | 6-8 | 6-7 | 7 | 8 | 11 | 16 | 16 | | | | | |
| 17 | 17 | 17 | 12 | 9 | 7 | 6-7 | 6-8 | 5-9 | 5-9 | 6-8 | 6-7 | 7 | 9 | 12 | 17 | 17 | | | | |
| 18 | 18 | 18 | 12 | 9 | 8 | 7 | 6-8 | 6-9 | [5-10] | 6-9 | 6-8 | 7 | 8 | 9 | 12 | 18 | 18 | | | |
| 19 | 19 | 19 | 13 | 10 | 8 | 7 | 6-8 | 6-9 | 5-11 | 5-11 | 6-9 | 6-8 | 7 | 8 | 10 | 13 | 19 | 19 | | |
| 20 | 20 | 20 | 14 | 10 | 8 | 7-8 | 6-8 | 6-9 | 6-10 | [5-8] | 6-10 | 6-9 | 6-8 | 7-8 | 8 | 10 | 14 | 20 | 20 | |
| 21 | 21 | 21 | 14 | 11 | 9 | 7 | 7-8 | 6-9 | 6-10 | 5-10 | 5-10 | 6-10 | 6-9 | 7-8 | 7 | 9 | 11 | 14 | 21 | 21 |
| 22 | 22 | 22 | 15 | 11 | 9 | 8 | 7-8 | 6-9 | 6-10 | 6-11 | [5-12] | 6-11 | 6-10 | 6-9 | 7-8 | 8 | 9 | 11 | 15 | 22 |
| 23 | 23 | 23 | 16 | 12 | 10 | 8 | 7-8 | 6-9 | 6-10 | 6-11 | 5-13 | 5-13 | 6-11 | 6-10 | 6-9 | 7-8 | 8 | 10 | 12 | 16 |
| 24 | 24 | 24 | 16 | 12 | 10 | 8-9 | 7-8 | 7-9 | 6-10 | 6-11 | 6-12 | [5-8] | 6-12 | 6-11 | 6-10 | 7-9 | 7-8 | 8-9 | 10 | 12 |
| 25 | 25 | 25 | 17 | 13 | 10 | 9 | 8 | 7-9 | 6-10 | 6-11 | 6-12 | 5-10 | 5-10 | 6-12 | 6-11 | 6-10 | 7-9 | 8 | 9 | 10 |
| 26 | 26 | 26 | 18 | 13 | 11 | 9 | 8 | 7-9 | 6-10 | 6-11 | 6-12 | 6-11 | [5-12] | 6-11 | 6-12 | 6-11 | 6-10 | 7-9 | 8 | 9 |
| 27 | 27 | 27 | 18 | 14 | 11 | 9-10 | 8-9 | 7-9 | 7-10 | 6-11 | 6-12 | 6-13 | 5-13 | 5-13 | 6-13 | 6-12 | 6-11 | 7-10 | 7-9 | 8-9 |
| 28 | 28 | 28 | 19 | 14 | 12 | 10 | 8 | 8-9 | 7-10 | 6-11 | 6-12 | 6-13 | 6-14 | [5-10] | 6-14 | 6-13 | 6-12 | 6-11 | 7-10 | 8-9 |
| 29 | 29 | 29 | 20 | 15 | 12 | 10-11 | 9 | 8-9 | 7-10 | 6-13 | 6-12 | 6-13 | 6-14 | 5-12 | 5-12 | 6-14 | 6-13 | 6-12 | 6-13 | 7-10 |
| 30 | 30 | 30 | 20 | 15 | 12 | 10 | 9 | 8-9 | 7-10 | 7-11 | 6-12 | 6-13 | 6-14 | 6-13 | [5-14] | 6-13 | 6-14 | 6-13 | 6-12 | 7-11 |
| 31 | 31 | 31 | 21 | 16 | 13 | 11 | 9-10 | 8-9 | 7-10 | 7-11 | 6-12 | 6-13 | 6-14 | 6-15 | 5-15 | 5-15 | 6-15 | 6-14 | 6-13 | 6-12 |
| 32 | 32 | 32 | 22 | 16 | 13 | 11 | 10 | 9 | 8-10 | 7-11 | 6-12 | 6-13 | 6-14 | 6-15 | 6-16 | [5-9] | 6-16 | 6-15 | 6-14 | 6-13 |
| 33 | 33 | 33 | 22 | 17 | 14 | 11 | 10 | 9 | 8-10 | 7-11 | 7-12 | 6-13 | 6-14 | 6-15 | 6-16 | 5-11 | 5-11 | 6-16 | 6-15 | 6-14 |
| 34 | 34 | 34 | 23 | 17 | 14 | 12 | 10-11 | 9 | 8-10 | 7-11 | 7-12 | 6-13 | 6-14 | 6-15 | 6-16 | 6-12 | [5-13] | 6-12 | 6-16 | 6-15 |
| 35 | 35 | 35 | 24 | 18 | 14 | 12 | 10-11 | 9-10 | 8-10 | 8-11 | 7-12 | 6-14 | 6-14 | 6-15 | 6-16 | 6-14 | 5-14 | 5-14 | 6-14 | 6-16 |
| 36 | 36 | 36 | 24 | 18 | 15 | 12 | 11-12 | 9 | 9-10 | 8-11 | 7-12 | 7-13 | 6-14 | 6-15 | 6-16 | 6-15 | 6-16 | [5-12] | 6-16 | 6-15 |
| 37 | 37 | 37 | 25 | 19 | 15 | 13 | 11 | 10 | 9-10 | 8-11 | 7-12 | 7-13 | 6-14 | 6-15 | 6-16 | 6-17 | 6-17 | 5-14 | 5-14 | 6-17 |
| 38 | 38 | 38 | 26 | 19 | 16 | 13 | 11-12 | 10 | 9-10 | 8-11 | 7-12 | 7-13 | 6-14 | 6-15 | 6-16 | 6-17 | 6-18 | 6-15 | [5-16] | 6-15 |
| 39 | 39 | 39 | 26 | 20 | 16 | 13 | 12 | 10-11 | 9-10 | 8-11 | 8-12 | 7-13 | 7-14 | 6-15 | 6-16 | 6-17 | 6-18 | 6-17 | 5-17 | 5-17 |
| 40 | 40 | 40 | 27 | 20 | 16 | 14 | 12 | 10-11 | 9-10 | 9-11 | 8-14 | 7-13 | 7-14 | 6-15 | 6-16 | 6-17 | 6-18 | 6-18 | 6-19 | [5-10] |
| 41 | 41 | 41 | 28 | 21 | 17 | 14 | 12-13 | 11 | 10 | 9-11 | 8-12 | 7-13 | 7-14 | 6-15 | 6-15 | 6-17 | 6-18 | 6-19 | 6-20 | 5-12 |
| 42 | 42 | 42 | 28 | 21 | 17 | 14 | 12 | 11-12 | 10 | 9-11 | 8-12 | 8-13 | 7-14 | 7-15 | 6-14 | 6-17 | 6-18 | 6-19 | 6-20 | 6-13 |
| 43 | 43 | 43 | 29 | 22 | 18 | 15 | 13 | 11-12 | 10 | 9-11 | 8-12 | 8-13 | 7-14 | 7-15 | 6-15 | 6-17 | 6-18 | 6-19 | 6-20 | 6-15 |
| 44 | 44 | 44 | 30 | 22 | 18 | 15 | 13 | 11-13 | 10-11 | 9-11 | 9-12 | 8-13 | 7-14 | 7-15 | 6-16 | 6-16 | 6-18 | 6-19 | 6-20 | 6-16 |
| 45 | 45 | 45 | 30 | 23 | 18 | 15 | 13 | 12 | 10 | 10-11 | 9-12 | 8-13 | 7-14 | 7-15 | 7-16 | 6-16 | 6-17 | 6-19 | 6-20 | 6-18 |
| 46 | 46 | 46 | 31 | 23 | 19 | 16 | 14 | 12-13 | 11 | 10-11 | 9-12 | 8-13 | 8-14 | 7-15 | 7-16 | 6-16 | 6-16 | 6-19 | 6-20 | 6-19 |
| 47 | 47 | 47 | 32 | 24 | 19 | 16 | 14 | 12-13 | 11 | 10-11 | 9-12 | 8-13 | 8-14 | 7-15 | 7-16 | 6-17 | 6-16 | 6-19 | 6-20 | 6-21 |
| 48 | 48 | 48 | 32 | 24 | 20 | 16 | 14 | 12-13 | 11-12 | 10-11 | 9-12 | 9-13 | 8-16 | 7-15 | 7-16 | 7-17 | 6-16 | 6-18 | 6-20 | 6-21 |

**Lemma 13** *For* $n \geq 1$, $R(4n, 2n) \leq 2 + R(2n, n)$.

**Proof**: Assume that $\mathcal{C} = \{S_1, S_2, \ldots, S_m\}$ is a $(2n, n)$-separator, where $m = R(2n, n)$. Construct $\mathcal{A} = \{A_1, A_2, \ldots, A_m, A_{m+1}, A_{m+2}\}$ as follows. If $i \in [m]$, then $A_i = S_i \cup (S_i + 2n)$. The other two sets are formed by taking $A_{m+1} = [2n]$ and $A_{m+2} = [2n] + 2n$.

We now argue that $\mathcal{A}$ has the desired properties. If $p, q \in [2n]$, then if $S_i$ separates $p$ from $q$ then $A_i$ also separates $p$ from $q$. If $p, q \in [2n] + 2n$, then if $S_i$ separates $p - 2n$ from $q - 2n$ then $A_i$ separates $p$ from $q$. If $p \in [2n]$ and $q \in [2n] + 2n$, then $A_{m+1}$ separates $p$ from $q$. If $p \in [2n] + 2n$ and $q \in [2n]$, then $A_{m+2}$ separates $p$ from $q$. Thus, in all cases $p$ is separated from $q$, and so $\mathcal{A}$ is a $(4n, 2n)$-separator of size $2 + R(2n, n)$. $\square$

As a consequence we have, for example, that $R(12, 6) \leq 6$, $R(16, 8) \leq 7$, $R(20, 10) \leq 8$, $R(24, 12) \leq 8$ and $R(28, 14) \leq 10$.

**Corollary 14** $R(2^m, 2^{m-1}) \leq 2m$.

**Proof**: The proof is by induction, the base case $R(2, 1) = 2$ following from earlier results, and the inductive step following from Lemma 13. $\square$

Note that the bound of the previous lemma is not tight since, for example, $R(8, 4) = 5 < 2 \cdot 3$. However, it does suggest logarithmic growth along the $k = n/2$ diagonal. Our final lemma provides us with bounds on the rate of growth of $R(n, k)$ with $n$, for a fixed $k$.

**Lemma 15** *(i) If* $n \geq 2k - 2$, $1 < k < n$, *then* $R(n + 1, k) \leq 2 + R(n, k)$,
*(ii) If* $n \geq 2k - 3$, $1 < k < n$, *then* $R(n + 2, k) \leq 3 + R(n, k)$,
*(iii) If* $n \geq 3k - 6$, $1 < k < n$, *then* $R(n + 3, k) \leq 3 + R(n, k)$.

**Proof**: We start with a minimal $(n, k)$-separator $\mathcal{S}$. (i) Append to $\mathcal{S}$ the sets $\{1, \ldots, k - 1, n + 1\}$ and $\{k, \ldots, 2k - 2, n + 1\}$. (ii) Append to $\mathcal{S}$ the sets $\{1, \ldots, k - 1, n+1\}$, $\{1, \ldots, k-1, n+2\}$ and $\{k, \ldots, 2k-3, n+1, n+2\}$. (iii) Append to $\mathcal{S}$ the sets $\{1, \ldots, k-2, n+1, n+2\}$, $\{k-1, \ldots, 2k-4, n+2, n+3\}$ and $\{2k-3, \ldots, 3k-6, n+3, n+1\}$. $\square$

Table 1 contains a summary of the known results for values of $R(n, k)$. The values for $k = n/2$ have been bracketed to highlight the symmetry. Where exact values are not known the bounds on the value are given. Lower bounds come from Lemmas 3 and 4 and Theorem 9. Upper bounds come from repeated applications of the lemmas from Lemma 10 onwards. Some of the specific cases for $k = 5$ have been proved by arguments similar to those employed in Lemma 8, while some of the upper bounds come from known example $(n, k)$-separators.

# 3 Final Remarks

There are many variations on problems involving separation schemes. To mention one of these, there is the question of minimising the size of a separating system where each set $A$ in the system has $1 < k \leq |A| \leq m < n$. For more open problems, see Ramsay et al [5].

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