Gray Codes from Antimatroids

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Abstract

We show three main results concerning Hamiltonicity of graphs derived from antimatroids. These results provide Gray codes for the feasible sets and basic words of antimatroids.

For antimatroid $(E, \mathcal{F})$, let $J(\mathcal{F})$ denote the graph whose vertices are the sets of $\mathcal{F}$, where two vertices are adjacent if the corresponding sets differ by one element. Define $J(\mathcal{F}; k)$ to be the subgraph of $J(\mathcal{F})^2$ induced by the sets in $\mathcal{F}$ with exactly k elements. Both graphs $J(\mathcal{F})$ and $J(\mathcal{F}; k)$ are connected, and the former is bipartite.

We show that there is a Hamiltonian cycle in $J(\mathcal{F}) \times K_2$. As a consequence, the ideals of any poset $\mathcal{P}$ may be listed in such a way that successive ideals differ by at most two elements. We also show that $J(\mathcal{F}; k)$ has a Hamilton path if $(E, \mathcal{F})$ is the poset antimatroid of a series-parallel poset.

Similarly, we show that $G(\mathcal{L}) \times K_2$ is Hamiltonian, where $G(\mathcal{L})$ is the "basic word graph" of a language antimatroid $(E, \mathcal{L})$. This result was known previously for poset antimatroids.

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1 Introduction

Antimatroids provide a very general setting in which to consider problems of generating combinatorial objects. In the context of ordering and listing the feasible sets or feasible words of antimatroids, Hamiltonicity questions naturally arise in graphs that have these feasible objects as vertices. The purpose of this paper is to raise and answer some of these Hamiltonicity questions.

A motivation for studying Hamiltonicity properties is their connections with combinatorial Gray Codes, that is, lists of combinatorial objects in which successive objects differ by a small amount. Such Gray Codes often form the basis of efficient algorithms for generating the objects.

Some proofs rely on the connectedness of distributive lattices to prove statements of the form "for every pair of objects x and y (lattice elements), there is a sequence of elementary transformations (cover relations) that transform x into y". For example, Propp [13] uses such a technique to prove that any perfect matching in a planar graph (subject to a mild condition) may be transformed into any other by a sequence of "face rotations". Using our results, any such statement implies a statement of the form "there is a sequence of all the objects so that successive objects differ by one or two elementary transformations".

Antimatroids lend themselves to two different representations, ordered and unordered. The ordered version is a kind of language, and the unordered version is a kind of set system.

A *set system* over a finite ground set E is a pair $(E, \mathcal{F})$ where $\mathcal{F} \subseteq 2^E$. A set system is an *antimatroid* if the following axioms hold.

(A1) If $F \in \mathcal{F}$ with $F \neq \emptyset$, then there exists an $x \in F$ such that $F \setminus \{x\} \in \mathcal{F}$.

(A2) If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

A set system satisfying (A1) is said to be *accessible.*

A *language* over a finite alphabet E is a pair $(E, \mathcal{L})$ with $\mathcal{L} \subseteq E^*$, where E^* is the free monoid of *words* in the alphabet E. The length of $\alpha \in \mathcal{L}$ (i.e., the number of not necessarily distinct letters in α) is denoted $|\alpha|$. The *support* $\tilde{\alpha}$ of α is the set of letters of α. A word α is *simple* if it contains no repeated letter (i.e., if $|\alpha| = |\tilde{\alpha}|$). By $E^{(*)}$ we denote the set of words in E^* which are simple. A language $(E, \mathcal{L})$ is *simple* if every word in $\mathcal{L}$ is simple (i.e., if each word of $\mathcal{L}$ is in $E^{(*)}$). A simple language is a *language antimatroid* if the following axioms hold.

(L1) If $\alpha\beta \in \mathcal{L}$, then $\alpha \in \mathcal{L}$.

(L2) If $\alpha, \beta \in \mathcal{L}$ with $\tilde{\alpha} \not\subseteq \tilde{\beta}$, then there is an $x \in \tilde{\alpha}$ with $\beta x \in \mathcal{L}$.

The *basic words*, $Basic(\mathcal{L})$, of a language antimatroid are the words $\alpha \in \mathcal{L}$ of maximal length.

There is a one-to-one correspondence between antimatroid set systems and antimatroid languages. Given an antimatroid set system $\mathcal{F}$, let $\mathcal{L}(\mathcal{F})$ be the language given by

$$\mathcal{L}(\mathcal{F}) \stackrel{\text{def}}{=} \{x_1 x_2 \cdots x_k : \{x_1, x_2, \ldots, x_i\} \in \mathcal{F} \;\; \forall i, \; 1 \leq i \leq k\}.$$

Given the antimatroid language $\mathcal{L}$, let $\mathcal{F}(\mathcal{L}) = \{\tilde{\alpha} : \alpha \in \mathcal{L}\}$. It is easy to see that $\mathcal{F}(\mathcal{L}(\mathcal{F})) = \mathcal{F}$, and $\mathcal{L}(\mathcal{F}(\mathcal{L})) = \mathcal{L}$. Thus we may switch from one representation to the other when convenient. Every antimatroid is both a set system and a language, so in order to avoid confusion over which representation we are referring to, we will most often use $\mathcal{F}$ or $\mathcal{L}$ when we wish to treat the antimatroid as a set system or language, respectively.

There are several alternate characterizations of antimatroids; one that is quite useful is in terms of "alternative precedences". Let E be a finite set and H be a mapping that associates with each $x \in E$ a set system $H(x) \subseteq 2^{E \setminus \{x\}}$. The language

$$\mathcal{L}_H \stackrel{\text{def}}{=} \{x_1 x_2 \cdots x_k \in E^{(*)} \; : \; \forall i, \; 1 \leq i \leq k, \exists \; A \in H(x_i) \text{ s.t. } A \subseteq \{x_1, x_2, \ldots, x_{i-1}\}\}$$

is called an *alternative precedence language.* A finite simple language is a language antimatroid if and only if it is an alternative precedence language ([11],[4]).

An element a is an *atom* if $\{a\} \in \mathcal{F}$; or, equivalently, if $a \in \mathcal{L}$; or, equivalently, if $H(a) = \{\emptyset\}$. In order to simplify the notation we sometimes use $\mathcal{F}$ to denote the antimatroid $(E, \mathcal{F})$, with the understanding that $E = \cup F$, where the union is over all $F \in \mathcal{F}$. Similarly, we will use $\mathcal{L}$ to denote the language antimatroid $(E, \mathcal{L})$, with the understanding that $E = \cup\tilde{\alpha}$, where the union is over all $\alpha \in \mathcal{L}$.

If $\mathcal{F}$ is an antimatroid and $A \in \mathcal{F}$, then we may obtain two new antimatroids by *contraction* and *deletion* as follows.

$$\mathcal{F}/A \stackrel{\text{def}}{=} \{F : F \cap A = \emptyset \text{ and } F \cup A \in \mathcal{F}\}.$$

$$\mathcal{F} \setminus A \stackrel{\text{def}}{=} \{F \in \mathcal{F} : F \cap A = \emptyset\}.$$

Note that these two antimatroids induce a partition of the feasible sets of $\mathcal{F}$ if $|A| = 1$.

Perhaps the most important class of antimatroids are the *poset antimatroids.* The set system $(E, \mathcal{F})$ is a poset antimatroid if E is the set of elements of a poset $\mathcal{P}$ and $\mathcal{F}$

is the set of ideals of $\mathcal{P}$. The basic words of the corresponding language antimatroid are the linear extensions of $\mathcal{P}$.

There are two standard "sums" of antimatroids. Let $(E_1, \mathcal{F}_1)$ and $(E_2, \mathcal{F}_2)$ be antimatroids on distinct ground sets. The *direct sum* $\mathcal{F}_1 + \mathcal{F}_2$ is the antimatroid

$$\mathcal{F}_1 + \mathcal{F}_2 \stackrel{\text{def}}{=} \{X_1 \cup X_2 : X_1 \in \mathcal{F}_1 \text{ and } X_2 \in \mathcal{F}_2\}.$$

The *ordinal sum* $\mathcal{F}_1 \oplus \mathcal{F}_2$ is the antimatroid

$$\mathcal{F}_1 \oplus \mathcal{F}_2 \stackrel{\text{def}}{=} \mathcal{F}_1 \cup \{E_1 \cup X_2 : X_2 \in \mathcal{F}_2\}.$$

Observe that starting with isolated elements and then applying the operations of direct and ordinal sum results in poset antimatroids of series-parallel posets.

We now define the graphs whose Hamiltonicity properties are investigated in the following sections. For antimatroid $(E, \mathcal{F})$, let $J(\mathcal{F})$ denote the undirected covering graph of the sets of $\mathcal{F}$ ordered by inclusion.

The *square* of a graph G, is the graph G^2 whose vertex set is the same as G and such that two vertices are adjacent in G^2 if they are of distance one or two in G. Define $J(\mathcal{F}; k)$ to be the subgraph of $J(\mathcal{F})^2$ induced by the sets in $\mathcal{F}$ with exactly k elements. Both graphs $J(\mathcal{F})$ and $J(\mathcal{F}; k)$ are connected, and the former is bipartite. For some $\mathcal{F}$, $J(\mathcal{F})$ does not have a Hamilton path, even when the partite sets of $J(\mathcal{F})$ are of the same cardinality; for example, consider the lattice of ideals of the series-parallel poset $\mathbf{1}\oplus(\mathbf{1}+\mathbf{1})\oplus\mathbf{1}\oplus(\mathbf{1}+\mathbf{1})\oplus\mathbf{1}$. The graph $J(\mathcal{F}; k)$ can fail to have Hamilton cycle but we know of no example where $J(\mathcal{F}; k)$ fails to have a Hamilton path. In fact, $J(\mathcal{F}; k)$ can be a path. If $\mathcal{F}$ is the poset antimatroid of a poset consisting of two disjoint chains, both containing $n-1$ elements, then $J(\mathcal{F}; n)$ is a path of length n.

A classic result of Ehrlich [5] and Bitner, Ehrlich and Reingold [2] is that $J(\mathcal{P}; k)$ is Hamiltonian if $\mathcal{P}$ is the poset antimatroid of an antichain. We extend this result to series-parallel posets.

The *basic word graph* $G(\mathcal{L})$ of the antimatroid $\mathcal{L}$ is the graph whose vertex set is $Basic(\mathcal{L})$, and such that two basic words are adjacent if they differ by the transposition of two adjacent elements. We show that $G(\mathcal{L}) \times K_2$ is Hamiltonian and thus the existence of a Gray Code for basic words, where each word differs from the previous one in the listing by one or two adjacent transpositions. We give an algorithm for generating the basic words which, for some antimatroids, runs in constant amortized time. Similar questions for the graph of linear extensions were considered by Pruesse and/or Ruskey [18], [15], [16], [14] and Stachowiak [21]. The diameter of $G(\mathcal{L})$ was investigated in [10] and [27].

In Section 2 we show that there is a Hamiltonian cycle in $J(\mathcal{F}) \times K_2$. As a consequence, the ideals of any poset $\mathcal{P}$ may be listed in such a way that successive ideals differ by at most two elements. In section 3 we show that $J(\mathcal{F};k)$ has a Hamilton path if $(E,\mathcal{F})$ is the poset antimatroid of a series-parallel poset. In Section 4 we show that $G(\mathcal{L}) \times K_2$ is Hamiltonian.

2 The graph $J(\mathcal{F}) \times K_2$ is Hamiltonian

We consider graphs of the form $G \times K_2$; that is, the product of a graph G with an edge. In other words, there are two copies of G, which we denote as $+G$ and $-G$, with edges joining corresponding vertices in the two copies. Given a vertex v in G, denote the corresponding vertices in $G \times K_2$ as $+v$ and $-v$. If w is a vertex of $G \times K_2$ then by $\overline{w}$ we denote the corresponding vertex in the other copy of G. The following lemma was proven in [16].

Lemma 1 *If G is bipartite and $G \times K_2$ is Hamiltonian, then G^2 is Hamiltonian.*

Theorem 1 *For every antimatroid $\mathcal{F}$ and any atom x of $\mathcal{F}$, the graph $J(\mathcal{F}) \times K_2$ has a Hamilton cycle that includes the edges $[+\emptyset, -\emptyset]$ and $[+\emptyset, +x]$.*

Proof: Our proof is by induction on $n = |E|$. If $n = 1$ then $J(\mathcal{F}) \times K_2$ is the 4-cycle $+\emptyset, +\{x\}, -\{x\}, -\emptyset$, where $E = \{x\}$.

If $n > 1$, then let $\{x\} \in \mathcal{F}$ and consider the antimatroid $\mathcal{F}/x$. We consider two cases depending upon the number of singleton sets in $\mathcal{F}$.

If x is the only atom of $\mathcal{F}$, then inductively we may assume that there is a Hamilton cycle $+\emptyset = X_1, X_2, \ldots, X_p = -\emptyset$ in $J(\mathcal{F}/x)$. This may be extended to the Hamilton cycle

$$+\emptyset, X_1 \cup \{x\}, X_2 \cup \{x\}, \ldots, X_p \cup \{x\}, -\emptyset$$

in $J(\mathcal{F})$. Clearly, this cycle contains the edges $[+\emptyset, -\emptyset]$ and $[+\emptyset, +\{x\}]$.

If there are at least two distinct elements $x, y \in E$ such that $\{x\}, \{y\} \in \mathcal{F}$, then inductively there are Hamilton cycles

$$+\emptyset = X_1, X_2, \ldots, X_p = -\emptyset \text{ in } J(\mathcal{F}/x)$$

and

$$+\emptyset = Y_1, Y_2, \ldots, Y_q = -\emptyset \text{ in } J(\mathcal{F} \setminus x),$$

where $X_2 = Y_2 = \{y\}$. The following is thus a Hamilton cycle in $G(\mathcal{F})$.

$$Y_1, X_1 \cup \{x\}, X_p \cup \{x\}, X_{p-1} \cup \{x\}, \ldots, X_2 \cup \{x\}, Y_2, Y_3, \ldots, Y_{q-1}, Y_q$$

This cycle contains the edges $[+\emptyset, -\emptyset]$ and $[+\emptyset, +\{x\}]$. $\square$

Observe that the preceeding proof maintains the following invariant:

The Hamilton path in $J(\mathcal{F}) \times K_2$ has the property that any feasible set I between two signed feasible sets $\pm F$ and $\mp F$ on the path satisfies $F \subset I$.

Thus the list of feasible sets has the structure of a well-formed parentheses string with $|\mathcal{F}|$ types of parentheses.

Corollary 1 *For every antimatroid $(E, \mathcal{F})$, there is a listing $X_1, X_2, \ldots, X_{|\mathcal{F}|}$ of the feasible sets $\mathcal{F}$ such that successive sets (including the first and last) differ by at most two elements.*

Proof: Since $J(\mathcal{F})$ is bipartite, we may apply Lemma 1 to conclude that $J(\mathcal{F})^2$ is Hamiltonian. □

3 Feasible sets of fixed size

Recall that the *Hamming distance* between two bitstrings is the number of positions in which they differ. Regarding the bitstrings as the characteristic vectors of a set the Hamming distance D between two sets is $D(A, B) = |(\overline{A} \cup \overline{B}) \cap (A \cup B)|$. In other words, the Hamming distance of two sets is the number of elements which are in one set but not in the other.

We defined $J(\mathcal{P}; k)$ to be the subgraph of $J(\mathcal{P})^2$ induced by those vertices (feasible sets) with exactly k elements. In other words, $J(\mathcal{P}; k) = (V, E)$ where $V = \{A \in \mathcal{F} : |A| = k\}$ and $E = \{\{A, B\} : A, B \in V \text{ and } D(A, B) = 2\}$. The graph $J(\mathcal{F}; k)$ is not bipartite in general (for example, the graph $J(2^{\{1,2,\ldots,n\}}; 1)$ is the complete graph K_n), but it is always connected.

Lemma 2 *For every antimatroid $(E, \mathcal{F})$, the graph $J(\mathcal{F}; k)$ is connected.*

Proof: Our proof is by induction on $|E|$. The lemma is clearly true if $k = 0, 1$ or if $|E| = 0, 1$. So assume that $|E| > 1$ and $k > 1$.

Let x be an atom of $\mathcal{F}$. Inductively, $J(\mathcal{F} \setminus x; k)$ and $J(\mathcal{F}/x; k-1)$ are connected. Thus, the vertices of $J(\mathcal{F}; k)$ which contain x are connected, as are those which do not contain x. If $F \in \mathcal{F} \setminus x$ with $|F| = k$, then there is a $y \in E \setminus \{x\}$ such that $F \setminus \{y\} \in \mathcal{F} \setminus x$. By closure under union $F' = F \setminus \{y\} \cup \{x\} \in \mathcal{F}$; i.e., $F \setminus \{y\} \in \mathcal{F}/x$. Since F and F' are adjacent in $J(\mathcal{F}; k)$, all vertices in $J(\mathcal{F}; k)$ are connected. □

If H is a Hamilton path, then $first(H)$ and $last(H)$ have the obvious meanings of the first and last vertices of H, respectively.

Definition 1 *The sequence $H_0(\mathcal{P}), H_1(\mathcal{P}), \ldots, H_n(\mathcal{P})$ is a* complete set of Hamilton paths *for the poset $\mathcal{P}$ if*

1. *Each* $H_k(\mathcal{P})$ *is a Hamilton path in* $J(\mathcal{P}; k)$ *for* $k = 1, 2, \ldots, n$.

2. $D(\mathrm{first}(H_i), \mathrm{first}(H_{i+1})) = 1$ *for* $i = 0, 1, \ldots, n-1$.

3. $D(\mathrm{last}(H_i), \mathrm{last}(H_{i+1})) = 1$ *for* $i = 0, 1, \ldots, n-1$.

Theorem 2 *If* $\mathcal{Q}$ *and* $\mathcal{R}$ *both have complete sets of Hamilton paths then* $\mathcal{P} = \mathcal{Q} \oplus \mathcal{R}$ *has a complete set of Hamilton paths.*

Theorem 3 *If* $\mathcal{Q}$ *and* $\mathcal{R}$ *both have complete sets of Hamilton paths then* $\mathcal{P} = \mathcal{Q} + \mathcal{R}$ *has a complete set of Hamilton paths.*

Corollary 2 *Series-parallel posets have complete sets of Hamilton paths.*

Corollary 3 *Forest posets have complete sets of Hamilton paths.*

Proof (of Theorem 2): Let $p = |S(\mathcal{P})|$, $q = |S(\mathcal{Q})|$, $r = |S(\mathcal{R})|$, and thus $p = q + r$. Given the complete sets of Hamilton paths $H_k(\mathcal{Q})$ (for $k = 0, 1, \ldots, q$) and $H_k(\mathcal{R})$ (for $k = 0, 1, \ldots, r$), we need to show a complete set of Hamilton paths $H_k(\mathcal{P})$ (for $k = 0, 1, \ldots, p$). For $k \le q$ set $H_k(\mathcal{P}) = H_k(\mathcal{Q})$. For $k > q$ set $H_k(\mathcal{P}) = S(\mathcal{Q}) \cup H_{k-q}(\mathcal{R})$, where the union is taken componentwise. □

Proof (of Theorem 3): Let $p = |S(\mathcal{P})|$, $q = |S(\mathcal{Q})|$, $r = |S(\mathcal{R})|$, and assume without loss of generality that $0 < r \le q < p$. The complete set of Hamilton paths for $\mathcal{Q}$ is denoted $H_k(\mathcal{Q})$ (for $k = 0, 1, \ldots, q$), and similarly $H_k(\mathcal{R})$ for $\mathcal{R}$. We will show how to construct a complete set of Hamilton paths $H_k(\mathcal{P})$ (for $k = 0, 1, \ldots, p$).

The global structure of the path for $H_k(\mathcal{P})$ consists of, for increasing values of i, Hamilton paths $p_{i,k-i}$ in the subgraph $H_i(\mathcal{Q}) \times H_{k-i}(\mathcal{R})$ of $J(\mathcal{P}; k)$.

In the *forward direction*, the $(i, k-i)$ path starts with

$$first(H_i(\mathcal{Q})) \cup first(H_{k-i}(\mathcal{R})) \quad \text{or} \quad first(H_i(\mathcal{Q})) \cup last(H_{k-i}(\mathcal{R})), \tag{1}$$

and ends at

$$last(H_i(\mathcal{Q})) \cup first(H_{k-i}(\mathcal{R})) \quad \text{or} \quad last(H_i(\mathcal{Q})) \cup last(H_{k-i}(\mathcal{R})). \tag{2}$$

To be specific, let

$$H_i(\mathcal{Q}) = a_1, a_2, \ldots, a_M \quad \text{and} \quad H_{k-i}(\mathcal{R}) = b_1, b_2, \ldots, b_N$$

Starting in the forward direction from (1), the Hamilton path in the subgraph $J(\mathcal{Q};i)\times J(\mathcal{R};k-i)$ starts out

$$\begin{array}{lcl} a_1b_1, a_1b_2, \ldots, a_1b_N, & & a_1b_N, a_1b_{N-1}, \ldots, a_1b_1, \\ a_2b_N, \ldots, a_2b_2, a_2b_1, & & a_2b_1, \ldots, a_2b_{N-1}, a_2b_N, \\ a_3b_1, a_3b_2, \ldots, a_3b_N, & \text{or} & a_3b_N, a_3b_{N-1}, \ldots, a_3b_1, \\ a_4b_N, \ldots, a_4b_2, a_4b_1, & & a_4b_1, \ldots, a_4b_{N-1}, a_4b_N, \\ \ldots & & \ldots \end{array}$$

and if M is odd ends at

$$a_Mb_1, a_Mb_2, \ldots, a_Mb_N \quad \text{or} \quad a_Mb_N, a_Mb_{N-1}, \ldots, a_Mb_1,$$

and if M is even ends at

$$a_Mb_N, \ldots, a_Mb_2, a_Mb_1 \quad \text{or} \quad a_Mb_1, a_1b_2, \ldots, a_Mb_N.$$

On this list successive feasible sets have the form a_sb_t, a_sb_{t+1} or $a_sb_t, a_{s+1}b_t$ and thus have Hamming distance 2. Thus for given values of i and k, there are two alternative lists. These two lists can be distinguished by their first elements. A $(i, k-i)$-list $p_{i,k-i}$ is *first-first* if it begins at a_1b_1 (i.e., $first(H_i(\mathcal{Q})) \cup first(H_{k-i}(\mathcal{R}))$; it is *first-last* if it begins at a_1b_N. The lists can also be distinguished by their last elements. A $(i, k-i)$-list is *last-last* if it ends at a_Mb_N; it is *last-first* if it ends at a_Mb_1. If M is odd, a first-first path is also last-last; otherwise, it is last-first.

We form the path $H_k(\mathcal{P})$ by stringing together paths $p_{i,k-i}$ for all valid values of i, where each $p_{i,k-i}$ is a path of one of the two types described above, and may appear either in the forward direction or in reverse; the reverse direction will be denoted with the superscript R. For example, when $k \le r \le q$,

$$H_k(\mathcal{P}) = \begin{cases} p_{k,0}, (p_{k-1,1})^R, \ldots, p_{0,k} & \text{if } k \text{ even,} \\ p_{k,0}, (p_{k-1,1})^R, \ldots, (p_{0,k})^R & \text{if } k \text{ odd.} \end{cases}$$

In order for $H_k(\mathcal{P})$ to be a path in $J(\mathcal{P}, k)$, the type of each $p_{i,k-i}$ (i.e., first-first or first-last) must be selected so that there is Hamming distance 2 in going from one path to the next. For example if $p_{i,k-i}$ is a first-first path and appears in reverse order in $H_k(\mathcal{P})$, then $p_{i-1,k-i+1}$ must be a first-first path. Note that the $p_{i,k-i}$'s are not necessarily all of the same type for a given k, because the number M of ideals of size i of $\mathcal{Q}$ may be even for some values of i, odd for others.

Observe that for any k and i, the type of $p_{i,k-i}$ can be selected, and that selection determines the type of $p_{j,k-j}$ for all other valid values of j.

Thus we have shown how to construct a set of paths $H_k(\mathcal{P})$ with Property 1; i.e., such that each $H_k(\mathcal{P})$ is a Hamilton path in $J(\mathcal{P}, k)$ for $k = 0, 1, \ldots, n$. We now show inductively how to select the types for the paths $p_{i,k-i}$ in $H_k(\mathcal{P})$ so that Properties 2 and 3 are maintained.

Initially, $H_0(\mathcal{P})$ is the path on one vertex, the empty set. Suppose the paths $H_0(\mathcal{P}), H_1(\mathcal{P}), \ldots, H_k(\mathcal{P})$ have been constructed so as to satisfy all three properties. Then $H_{k+1}(\mathcal{P})$ can be constructed so as to have Properties 2 and 3 in the following way.

[Case where $k < r \leq q$:] Assume for the sake of illustration that k is even; similar arguments hold when k is odd. Then

$$\begin{aligned} H_k(\mathcal{P}) = &\quad p_{k,0}, (p_{k-1,1})^R, \ldots, p_{0,k}, \\ H_{k+1}(\mathcal{P}) = &\quad p_{k+1,0}, (p_{k,1})^R, \ldots, (p_{0,k+1})^R. \end{aligned}$$

Note that, regardless of type, $D(first(p_{k,0}), first(p_{k+1,0})) = 1$. If the type of $p_{0,k}$ is last-last, then set the type of $p_{0,k+1}$ to be first-last; otherwise, set it to be first-first. $H_0(\mathcal{Q})$ just contains the empty set, so $a_1 = a_M$, and $D(first(H_0(\mathcal{Q})), last(H_0(\mathcal{Q}))) = 0$. By definition,

$$D(last(H_k(\mathcal{R})), last(H_{k+1}(\mathcal{R}))) = 1.$$

Therefore $D(last(H_k(\mathcal{P})), last(H_{k+1}(\mathcal{P}))) = 1$.

[Case where $r \leq k < q$:] In this case, the number of paths strung together is $r + 1$. Suppose r is even. Then

$$\begin{aligned} H_k(\mathcal{P}) = &\quad p_{k,0}, (p_{k-1,1})^R, \ldots, p_{k-r,r}, \\ H_{k+1}(\mathcal{P}) = &\quad p_{k+1,0}, (p_{k,1})^R, \ldots, p_{k+1-r,r}. \end{aligned}$$

By a similar argument as given in the first case,

$$D(first(H_k(\mathcal{P})), first(H_{k+1}(\mathcal{P}))) = D(last(H_k(\mathcal{P})), last(H_{k+1}(\mathcal{P}))) = 1,$$

regardless of type. A similar argument holds when r is odd.

[Case $r \leq q < k$:] For $q + r - k$ even,

$$\begin{aligned} H_k(\mathcal{P}) = &\quad p_{q,k-q}, (p_{q-1,k+1-q})^R, \ldots, p_{k-r,r}, \\ H_{k+1}(\mathcal{P}) = &\quad (p_{q,k+1-q})^R, p_{q-1,k+2-r}, \ldots, p_{k+1-r,r}. \end{aligned}$$

If $p_{q,k-q}$ is first-first, set $p_{q,k+1-q}$ to be last-first. It is easy to confirm that Properties 2 and 3 are upheld. A similar argument holds when $q + r - k$ is odd.

This completes the proof. $\square$

4 The graph $G(\mathcal{L}) \times K_2$ is Hamiltonian

In this section, we show the existence of a certain kind of Gray code for the basic words of an antimatroid. We begin with a definition and two lemmas.

Definition 2 *For a word α, let $\alpha - x$ be the word $\beta\gamma$, where $\alpha = \beta x \gamma$, or $\alpha - x = \alpha$, if $x \notin \alpha$.*

Note that if x is an atom that $\mathcal{L}/\{x\} = \{\alpha - x : \alpha \in \mathcal{L}\}$.

Lemma 3 *Let $(E, \mathcal{L})$ be an antimatroid in which the elements of E are $\{a, x_1, x_2, \ldots, x_n\}$ where a is an atom. For all $1 \leq i \leq j < n$, if*

$$\pi = x_1 \cdots x_i x_{i+1} \cdots x_j a x_{j+1} \cdots x_n \ \in \ \mathcal{L},$$

then

$$\pi' = x_1 \cdots x_i a x_{i+1} \cdots x_j x_{j+1} \cdots x_n \ \in \ \mathcal{L}.$$

Proof: Let H be a alternative precedence mapping for $\mathcal{L}$. Since a is an atom, $H(a) = \{\emptyset\}$ and so a needs no precedent set. For the x_l $(1 \leq l \leq n)$ elements, it is clear that the same precedent constraints met in π are also met in π'. $\square$

Definition 3 *A* B-poset *is a poset $\mathcal{P}$ whose elements can be partitioned into two disjoint chains, $x_1 \prec x_2 \prec \cdots \prec x_n$ and $y_1 \prec y_2 \prec \cdots \prec y_m$, such that $y_j \not\preceq x_i$ for all i and j, where $1 \leq i \leq n$ and $1 \leq j \leq m$.*

The following lemma was proved in [15].

Lemma 4 *Let $\mathcal{P}$ be a B-poset as defined above and denote $\kappa = x_1 x_2 \cdots x_n y_1 y_2 \cdots y_m$. There is a Hamiltonian cycle in $G(\mathcal{P}) \times K_2$ which uses the edge $[+\kappa, -\kappa]$.*

We now give the main theorem of this section. The theorem and proof given here are generalizations of the main theorem and proof in [16], which showed the following result for poset antimatroids.

Theorem 4 *For every antimatroid $\mathcal{L}$, the graph $G(\mathcal{L}) \times K_2$ is Hamiltonian.*

Proof: The proof of the theorem is by induction on $|E|$. For the base cases of the induction, $\mathcal{L}$ is the antimatroid on zero or one elements; in these cases $G(\mathcal{F}) \times K_2$ is an edge.

Suppose $|E| > 1$. If $\mathcal{L}$ has exactly one atom a, then $G(\mathcal{L}) \cong G(\mathcal{L}/\{a\})$, and by the inductive hypothesis $G(\mathcal{L}/\{a\}) \times K_2$ is Hamiltonian.

Otherwise, let $\mathcal{L}$ have two atoms a and b. By the inductive hypothesis, the graph $G(\mathcal{L}/\{a,b\}) \times K_2$ has a Hamiltonian cycle H_1. Replace each signed basic word $+\alpha_i$ on H_1 with $ab\alpha_i$; replace each basic word $-\alpha_i$ with $ba\alpha_i$. Call the resulting sequence H_2. By Lemma 3, both $ab\alpha_i$ and $ba\alpha_i$ are basic words of $\mathcal{L}$; therefore H_2 is a cycle $\beta_1, \beta_2, \ldots, \beta_M$, where $M = 2 \cdot e(\mathcal{L}/\{a,b\})$, in $G(\mathcal{L})$, which visits exactly those basic words in $\mathcal{L}$ in which have the prefix ab or ba.

For each $\beta_i = x_i y_i \zeta_i$, where $\{x_i, y_i\} = \{a, b\}$, let B_i be the poset which consists of the two chains, $x_i y_i$ and ζ_i, plus the relations given (directly or transitively) by the following rule. For $z \in \tilde{\zeta_i}$, where σ and ρ are such that $\zeta_i = \sigma z \rho$, if $x_i \sigma z y_i \rho \notin \mathcal{L}$ then $y_i <_{B_i} z$; similarly, if $\sigma z x_i y_i \rho \notin \mathcal{L}$ then $x_i <_{B_i} z$.

Consider a linear extension $\omega = \delta x_i \gamma y_i \epsilon$ of B_i. By the construction of B_i, $x_i \delta \gamma y_i \epsilon \in \mathcal{L}$, and $\delta x_i y_i \gamma \epsilon \in \mathcal{L}$. By Lemma 3, $\omega \in \mathcal{L}$. Therefore every linear extension of B_i is a basic word of $\mathcal{L}$.

We observe also that B_i is a B-poset. Therefore by Lemma 4, there is a Hamiltonian path through $G(B_i) \times K_2$ from $+\beta_i$ to $-\beta_i$. We substitute the occurrence of β_i with this path in H_2, for each odd i. For each even i, we substitute the reverse of this path for the occurrence of β_i. Call the resulting walk H.

For each B_i, every linear extension of B_i is a basic word of $\mathcal{L}$; therefore H is a closed walk in $G(\mathcal{L}) \times K_2$. To prove that H is a Hamiltonian cycle through $G(\mathcal{L}) \times K_2$, it is necessary to show that for each basic word γ of $\mathcal{L}$, $+\gamma$ and $-\gamma$ both occur exactly once on H. Let $\zeta = \gamma - a - b$. Suppose γ induces the order xy on $\{a, b\}$ and the order ζ on $S(\mathcal{L}) \backslash \{a, b\}$. Then $xy\zeta$ is a basic word of $\mathcal{L}$, and therefore $xy\zeta = \beta_i$ is on the cycle H_2 of $G(\mathcal{L})$. Therefore γ is a linear extension of the B_i, and so both $+\gamma$ and $-\gamma$ are on the cycle H. Also, for every other B-poset B_j generated, where $j \neq i$, one of xy or ζ is not a subword of B_j, as in that case $\beta_i = \beta_j$, which contradicts the construction of H_2. Therefore, each of $+\gamma$ and $-\gamma$ are generated exactly once. $\square$

We call a pair a, b of elements of E *siblings* in a language $\mathcal{L}$ if for all basic words in $\mathcal{L}$, transposing a and b yields another word in $\mathcal{L}$. Let $\mathcal{L} + ab$ denote the language consisting of the words of $\mathcal{L}$ in which a precedes b. If $\mathcal{L}$ is an antimatroid, then so is $\mathcal{L} + ab$: we may simply add a to each $A \in H(b)$ where H is the alternative precedence function of $\mathcal{L}$, yielding an alternative precedence representation for $\mathcal{L} + ab$.

Let $G'(\mathcal{L})$ be the graph with the basic words of $\mathcal{L}$ as the vertex set, and such that two basic words are adjacent in $G'(\mathcal{L})$ if they differ by the transposition of two (not necessarily adjacent) elements. Thus $G'(\mathcal{L})$ is a supergraph of $G(\mathcal{L})$, on the same vertex set.

Corollary 4 *If $\mathcal{L}$ is an antimatroid with a pair of siblings, then $G'(\mathcal{L})$ is Hamiltonian.*

Proof: Suppose $\mathcal{L}$ has a pair of siblings a, b. By Theorem 4, $G(\mathcal{L} + ab) \times K_2$ is Hamiltonian; therefore $G'(\mathcal{L} + ab) \times K_2$ is Hamiltonian. Note that $G'(\mathcal{L}) \cong G'(\mathcal{L} + ab) \times K_2$. Therefore $G'(\mathcal{L})$ is Hamiltonian. $\square$

It is an open problem to determine whether $G(\mathcal{L})$ is Hamiltonian, where $\mathcal{L}$ is an antimatroid with a pair of siblings.

The graph $G(\mathcal{L})$ is bipartite; therefore applying Theorem 4 and Lemma 1, we conclude the following corollary, which establishes a Gray Code for basic words of an antimatroid.

Corollary 5 *For any antimatroid $\mathcal{L}$, there is a listing of the basic words of $\mathcal{L}$ such that each word differs from it predecessor in the list by one or two adjacent transpositions.*

This is the best possible, in that not all antimatroids admit a listing of the basic words such that each differs from its predecessor by one transposition. The series-parallel poset antimatroid $(\mathbf{1} \oplus \mathbf{1}) + (\mathbf{1}) \oplus \mathbf{1})$ provides such an example. In that case, the difference in the cardinality of the two partite sets of $G(\mathcal{L})$, which quantity we will call the *parity difference* of $\mathcal{L}$, exceeds one (i.e., there are two more odd basic words than even), so there can exist no Hamiltonian path or cycle in the basic word graph. The following conjecture of Ruskey [19] suggests a characterization of the posets which are generable by transpositions.

Conjecture 1 *For a poset $\mathcal{P}$, if the parity difference of $\mathcal{P}$ does not exceed one, then $G(\mathcal{P})$ has a Hamilton path; if the parity difference is zero and $G(\mathcal{P})$ has no vertex of degree one, then $G(\mathcal{P})$ is Hamiltonian.*

This conjecture cannot be extended to all antimatroids; an antimatroid with zero parity difference for which $G(\mathcal{L})$ has no Hamilton path is shown in Figure 1.

A generation algorithm runs in *constant amortized time* if its running time is $O(e)$, where e is the number of objects generated. An *O(f(n)) transposition oracle* of an antimatroid is an oracle which determines in time $O(f(n))$ whether $\alpha xy\beta$ is a basic word of $\mathcal{L}$, given that $\alpha yx\beta$ is a basic word of $\mathcal{L}$.

The algorithm `GenLE` and the proof of its correctness given in [16] to generate linear extensions uses only those properties of posets that are shared by all antimatroid languages. The queries to the antimatroid are of the type, "Is $\alpha xy\beta$ in $\mathcal{L}$?", where $\alpha yx\beta$ is in $\mathcal{L}$. Therefore, given an antimatroid with an $O(f(n))$ transposition oracle, `GenLE` is a $O(f(n))$ amortized time generation algorithm. In particular, we have the following result.

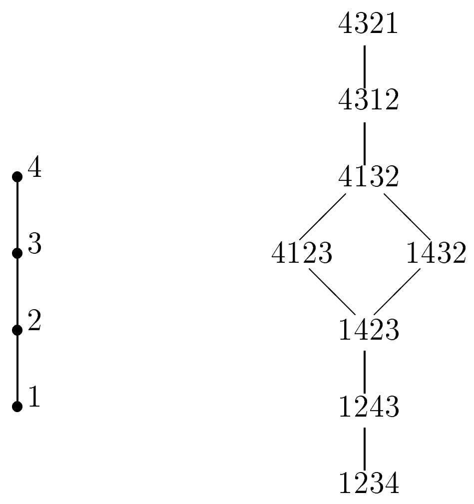

Figure 1: The path P_4, and $G(\mathcal{L})$, where $\mathcal{L}$ is the leaf shelling antimatroid for P_4.

Theorem 5 *Let $\mathcal{L}$ be an antimatroid with an $O(1)$ transposition oracle. Then the basic words of an antimatroid can be generated in constant amortized time, such that each word differs from the next by no more than two transpositions.*

Clearly poset antimatroids have $O(1)$ transposition oracles. Another example of an antimatroid with an $O(1)$ transposition oracle is given below.

For a tree G, the *leaf shelling sequence* of G is a sequence $v_1, v_2, \ldots, v_l$, where $v_i \in V(G)$, for all $1 \leq i \leq l$, and v_i is a leaf of the subgraph of G which remains when the vertices $\{v_1, v_2, \ldots, v_{i-1}\}$ are removed. I.e., these are ways of removing the vertices of G such that we only ever remove a leaf from the current tree. It is easy to verify that leaf shelling sequences form an antimatroid language. Suppose that $\alpha xy\beta$ is a leaf shelling sequence of G. If β is empty, or if x and y are non-adjacent, then $\alpha yx\beta$ is also a leaf shelling sequence; otherwise x and y are adjacent, and x and y are not that last two vertices to be removed, so removing y before removing x will disconnect x from the rest of the tree. Therefore $\alpha yx\beta$ is not a shelling sequence. Testing these conditions can be done in time $O(1)$, so by Theorem 5, the complete shelling sequences of a tree (i.e., the shelling sequences which contain all the vertices of G) can be generated in consant amortized time.

Chordal graphs admit a *decomposition sequence*; i.e., a sequence $v_i, v_2, \ldots, v_n$, where for each v_i, its neighbourhood in the graph G induced on $v_{i+1}, v_{i+2}, \ldots, v_n$ is a clique. The decomposition sequences of a chordal graph form an antimatroid language. However, a constant time transposition oracle is not known to exist for this language.

5 Final Remarks

The remarks at the end of the last section show that there are efficient (constant amortized time) algorithms for generating the basic words of an antimatroid; but what about the efficient generation of feasible sets and, in particular, the ideals of a poset? Algorithms for generating the ideals of a poset were given by Schrage and Baker [20], Lawler [12], Ball and Provan [1], and Steiner [22]. The best of these algorithms appears to be that of Steiner, whose running time is $O(n \cdot i(\mathcal{P}))$, where n is the number of elements of the poset and $i(\mathcal{P})$ is the number of ideals of $\mathcal{P}$. We have developed an algorithm based on the results of Section 2 whose running time is $O(i(\mathcal{P}))$, but only on certain posets; in the worst case it is also $O(n \cdot i(\mathcal{P}))$, so it remains an open problem to develop a constant amortized time algorithm for generating the ideals of a general poset. Beyer and Ruskey [3] and Koda and Ruskey [9] have shown that the ideals of a forest poset could be generated in time $O(i(\mathcal{P}))$; the latter paper contains a "loopfree" algorithm.

To end the paper we propose two conjectures.

Conjecture 2 *Let $\mathcal{P}$ be a poset and consider the subgraph G of $J(\mathcal{P})$ induced by the vertices of $J(\mathcal{P};k) \cup J(\mathcal{P};k+1)$. The graph G is bipartite, say with bipartition (A, B). If $|A| = |B|$ then is there a Hamilton cycle in G? If $|A| = |B| + 1$ then is there a Hamilton path in G? We conjecture that the answer is yes in both cases.*

Note that this conjecture contains the famous "middle two levels of the boolean algebra lattice problem" as a special case and so is most likely very difficult (e.g., Kierstead and Trotter [8] or Hurlbert [6]), if indeed it is true.

Conjecture 3 *For every antimatroid $(E, \mathcal{F})$ and every k, the graph $J(\mathcal{F};k)$ has a Hamilton path.*

This cannot be strengthened to Hamilton cycle since $J(\mathcal{F};k)$ is a path if $\mathcal{F}$ is the poset antimatroid of a poset consisting of two disjoint chains.

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