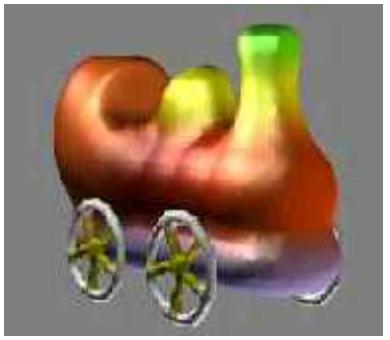


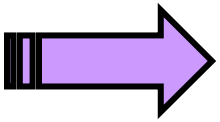


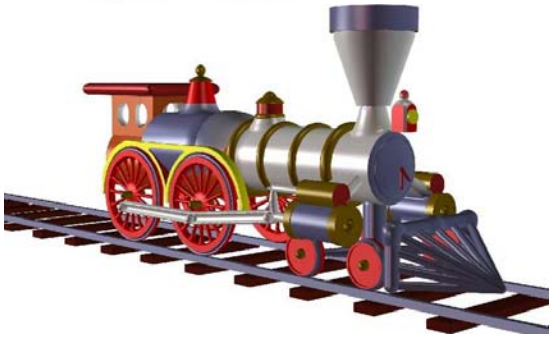


CSC 305
Transformations

by Brian Wyvill

The University of Victoria
Graphics Group





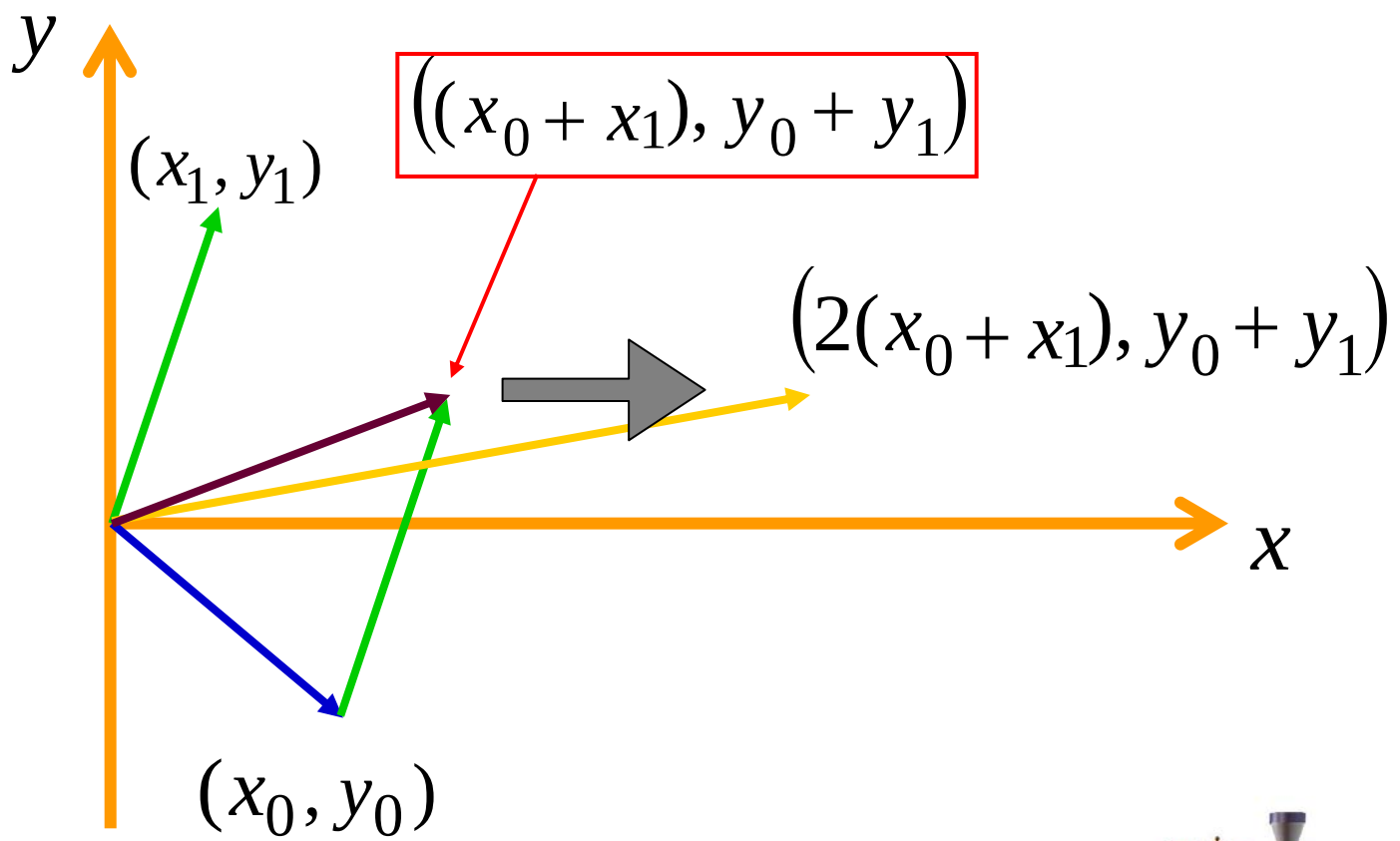
Transformations and Matrices

- Transformations are functions
- Matrices are function representations
- Matrices represent linear transforms
- {2x2 Matrices ≡ {2D Linear Transf's}



$$(2(x_0 + x_1), y_0 + y_1)$$

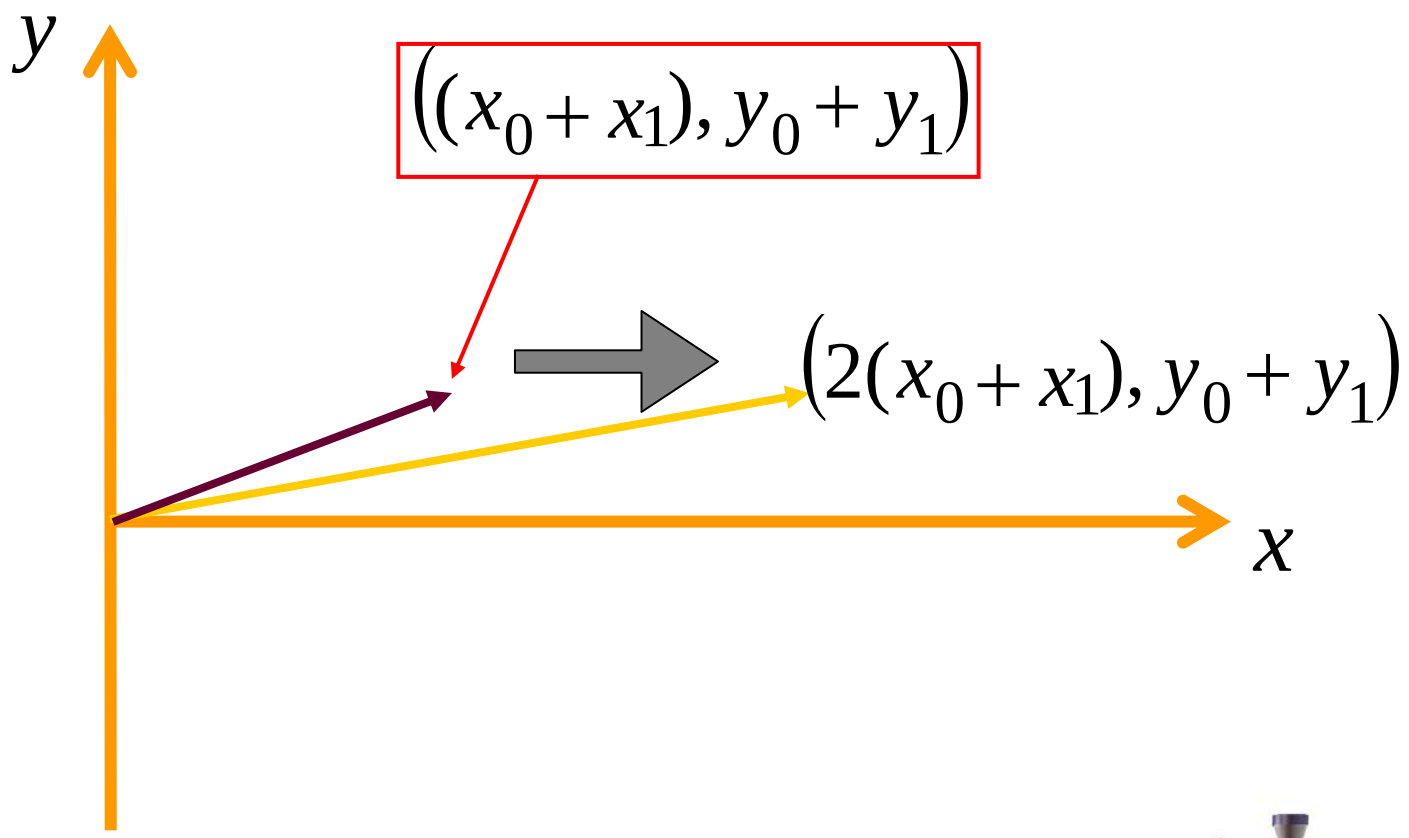
add then scale





$$(2(x_0 + x_1), y_0 + y_1)$$

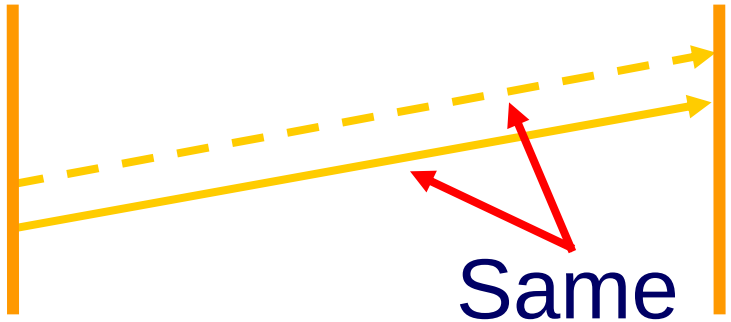
scale then add





Summary on Scale

- "*Scale* then *add*," is same as
- "*Add* then *scale*"





Matrix Representation

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$$



Matrix Representation of $S_{2_y}(\mathbf{v})$

Scale **in** ***y*** **by 2:** $S_{2_y}(v)$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2y \end{bmatrix}$$



Matrix Representation $S_2(v)$

***Overall* Scale by *2*: $S_2(v)$**

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$



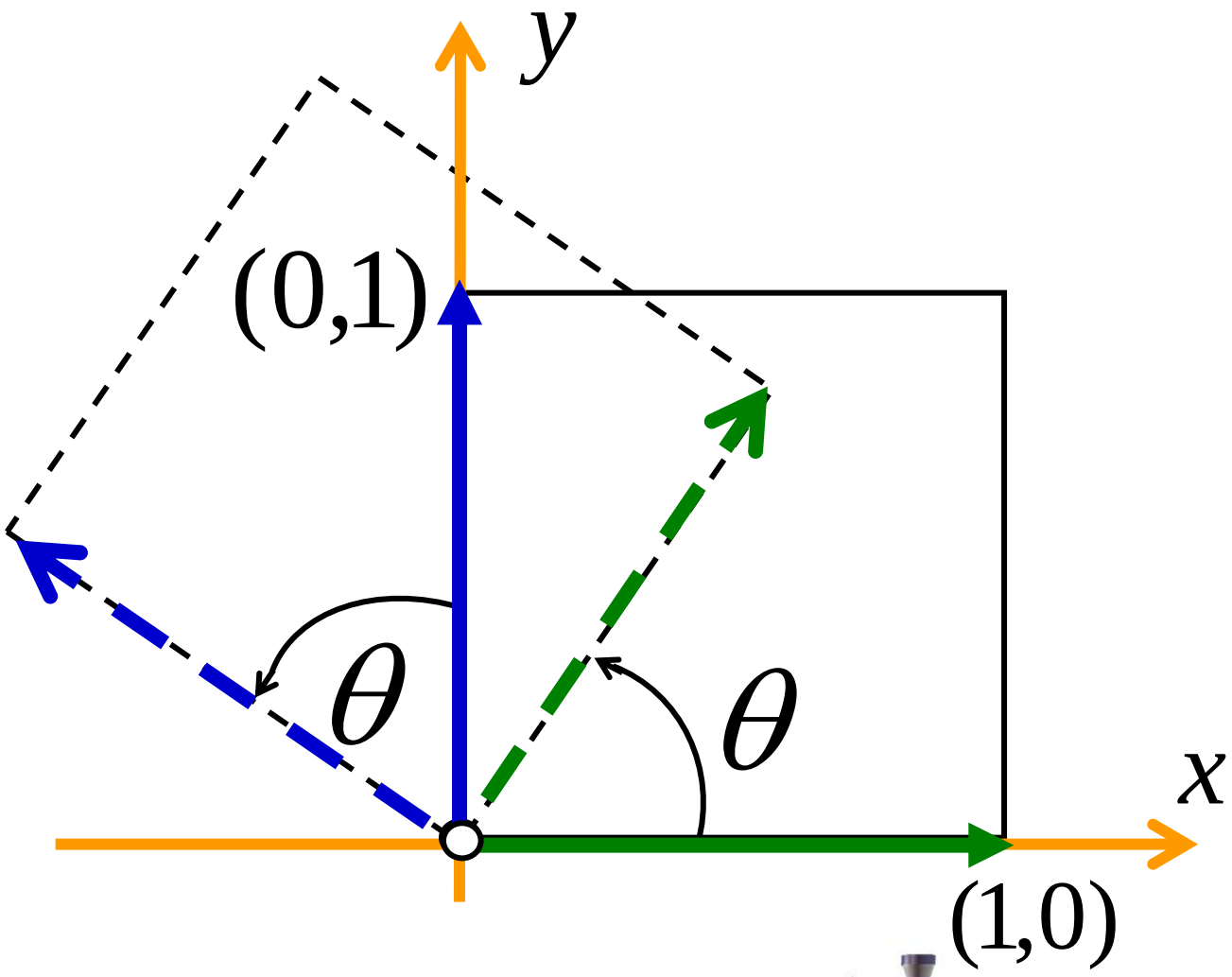
Matrix Form of Same

$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_0 + x_1 \\ y_0 + y_1 \end{bmatrix}}_{\text{Add } x \text{ and } y\text{, then scale}} = \begin{bmatrix} 2(x_0 + x_1) \\ y_0 + y_1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2x_0 + 2x_1 \\ y_0 + y_1 \end{bmatrix}}_{\text{Scale } x \text{ and } y\text{, then add}}$$

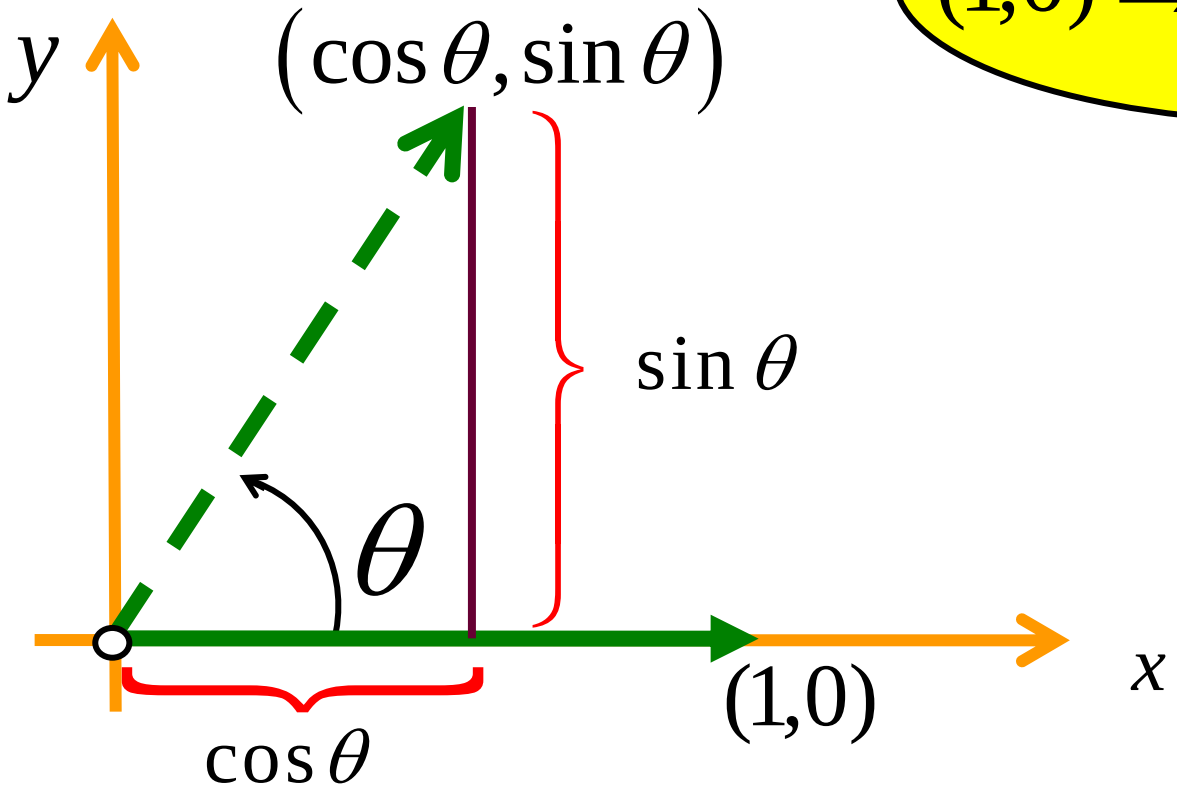


Rotate by θ



Rotate by θ: 1st Quadrant

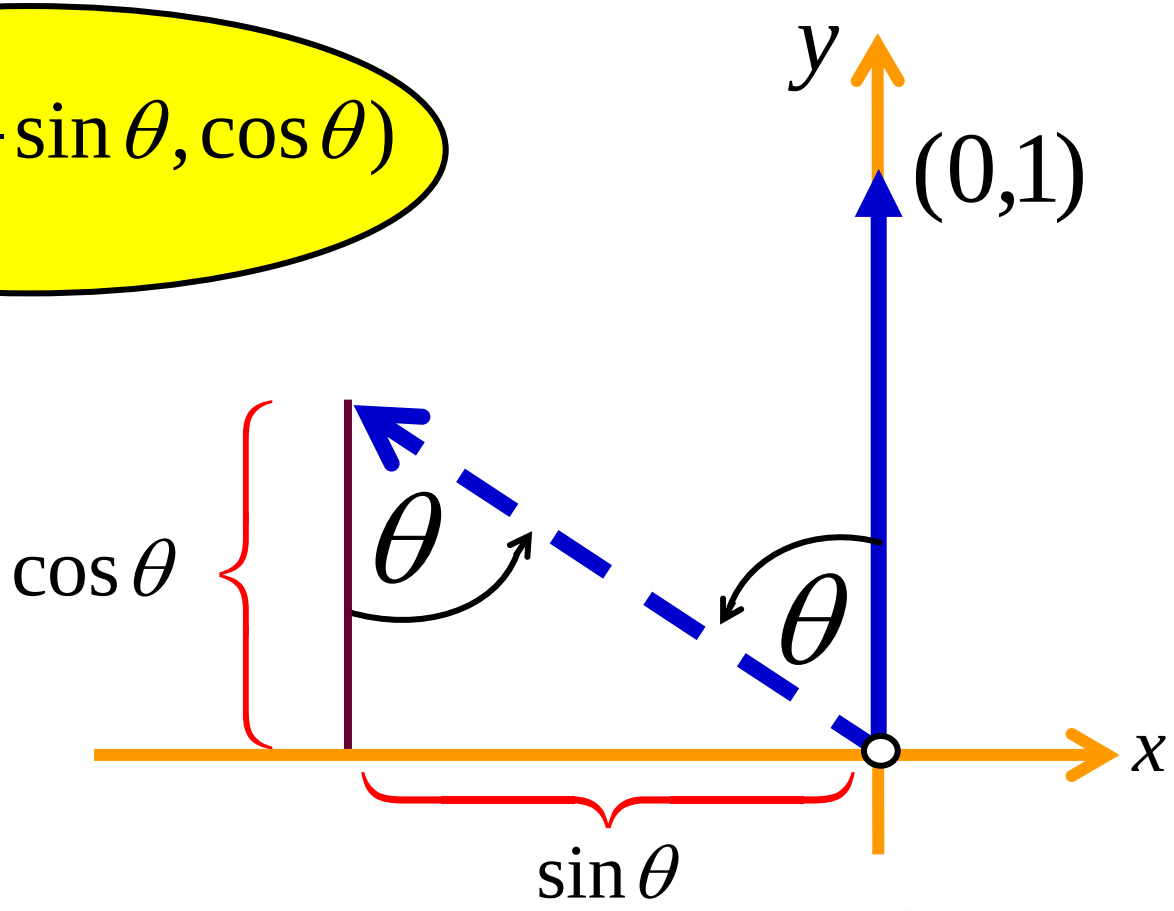
$(1,0) \Rightarrow (\cos\theta, \sin\theta)$





Rotate by θ: 2nd Quadrant

$$(0,1) \Rightarrow (-\sin\theta, \cos\theta)$$





Summary (Column Form)

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$



Using Matrix Notation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

(Note that *unit vectors* simply copy columns)



General Rotation by θ *Matrix*

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$



Who understands matrices?

What do the off diagonal elements do?

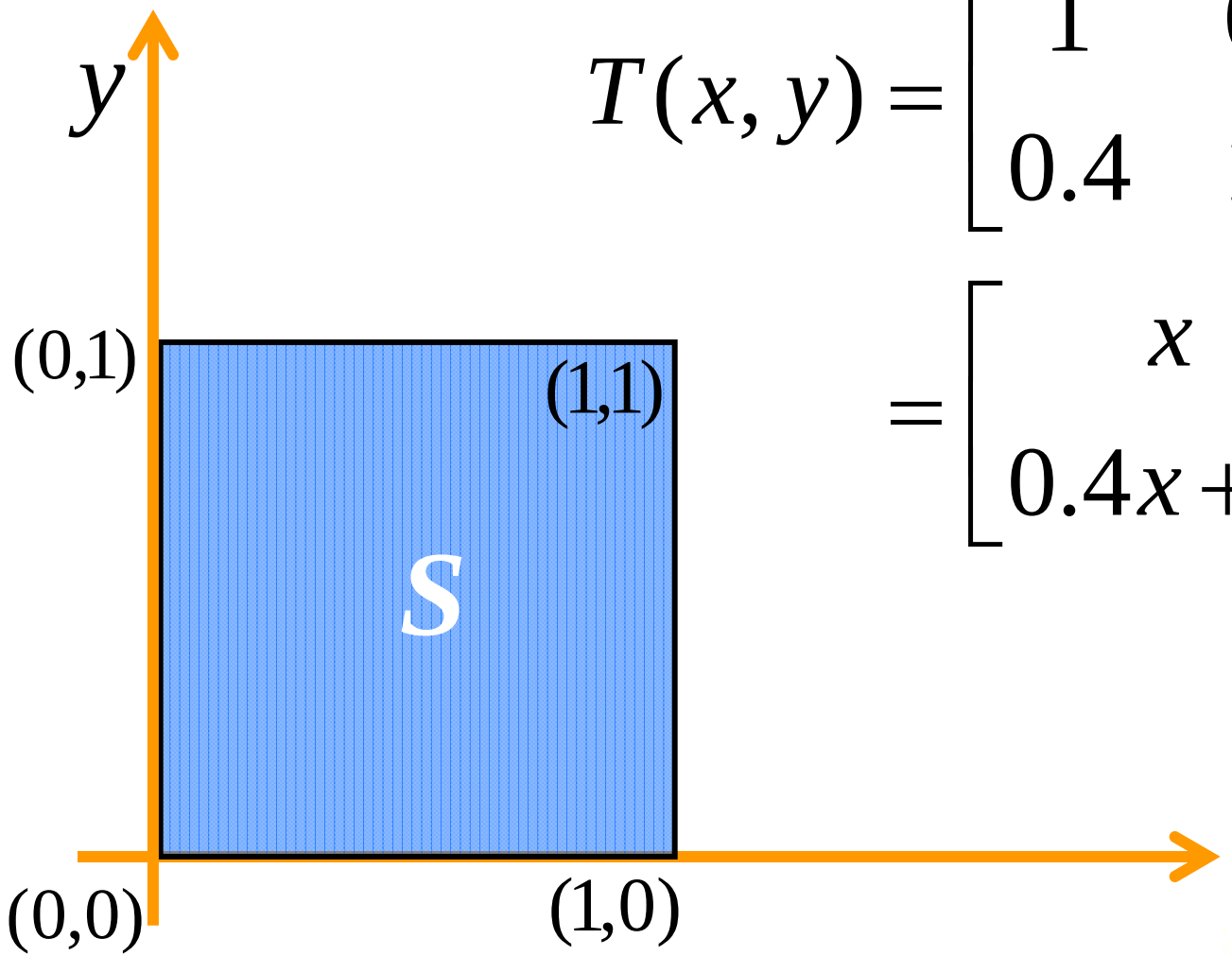


Off Diagonal Elements

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + a\,y \\ y \end{bmatrix}$$

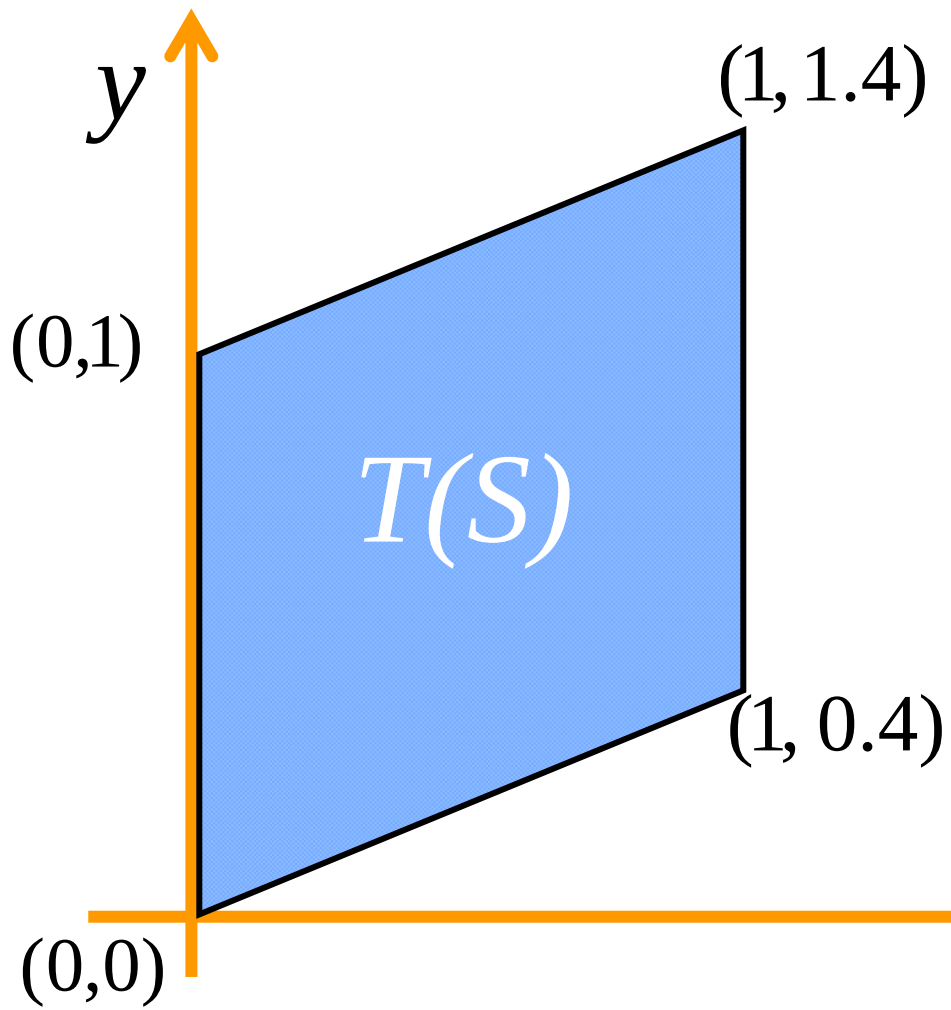
$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ b\,x + y \end{bmatrix}$$



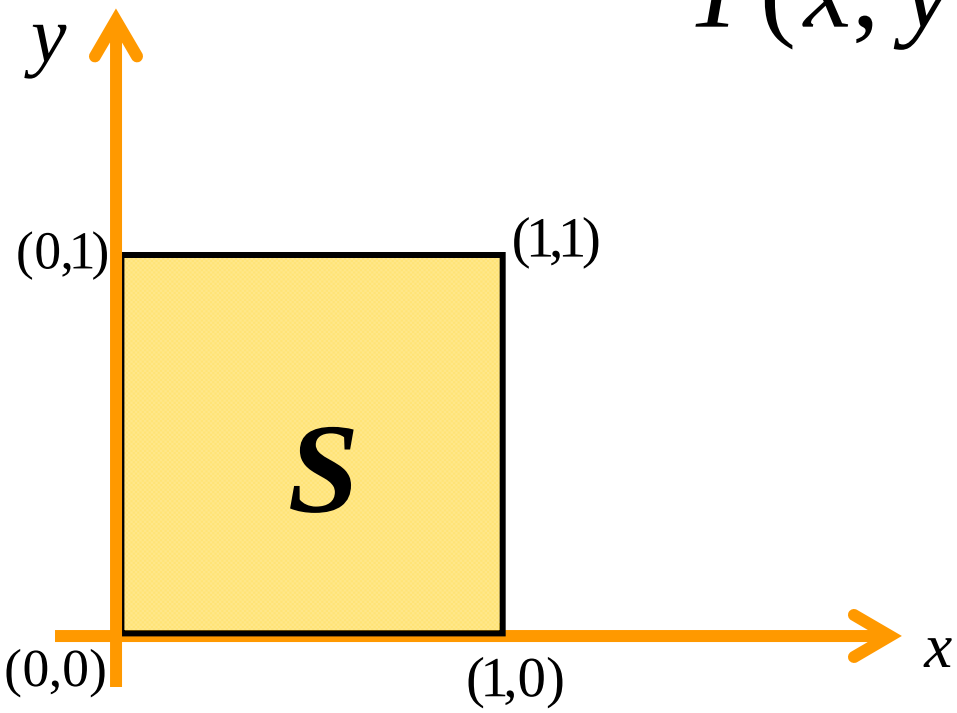


$$T(x,y) = \begin{bmatrix} 1 & 0 \\ 0.4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x \\ 0.4x + y \end{bmatrix}$$



$$T(x,y) = \begin{bmatrix} x \\ 0.4x + y \end{bmatrix}$$

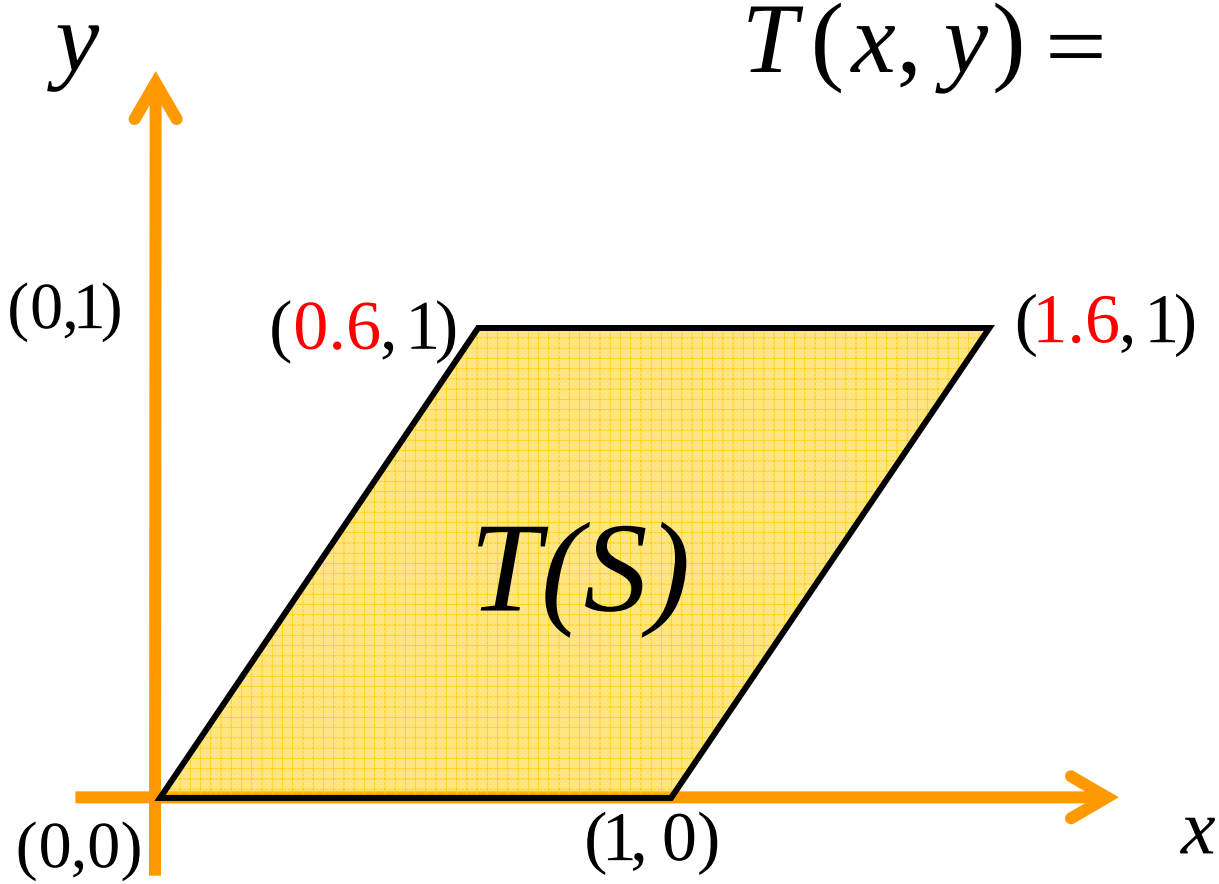


$$T(x,y) = \begin{bmatrix} 1 & 0.6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x + 0.6y \\ y \end{bmatrix}$$



$$T(x,y) = \begin{bmatrix} x + 0.6y \\ y \end{bmatrix}$$





Summary

$$Sh_x = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$$

$$Sh_y = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ bx + y \end{bmatrix}$$



Double Shear: not commutative!

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} = \begin{bmatrix} (1+ab) & a \\ b & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ b & (1+ab) \end{bmatrix}$$



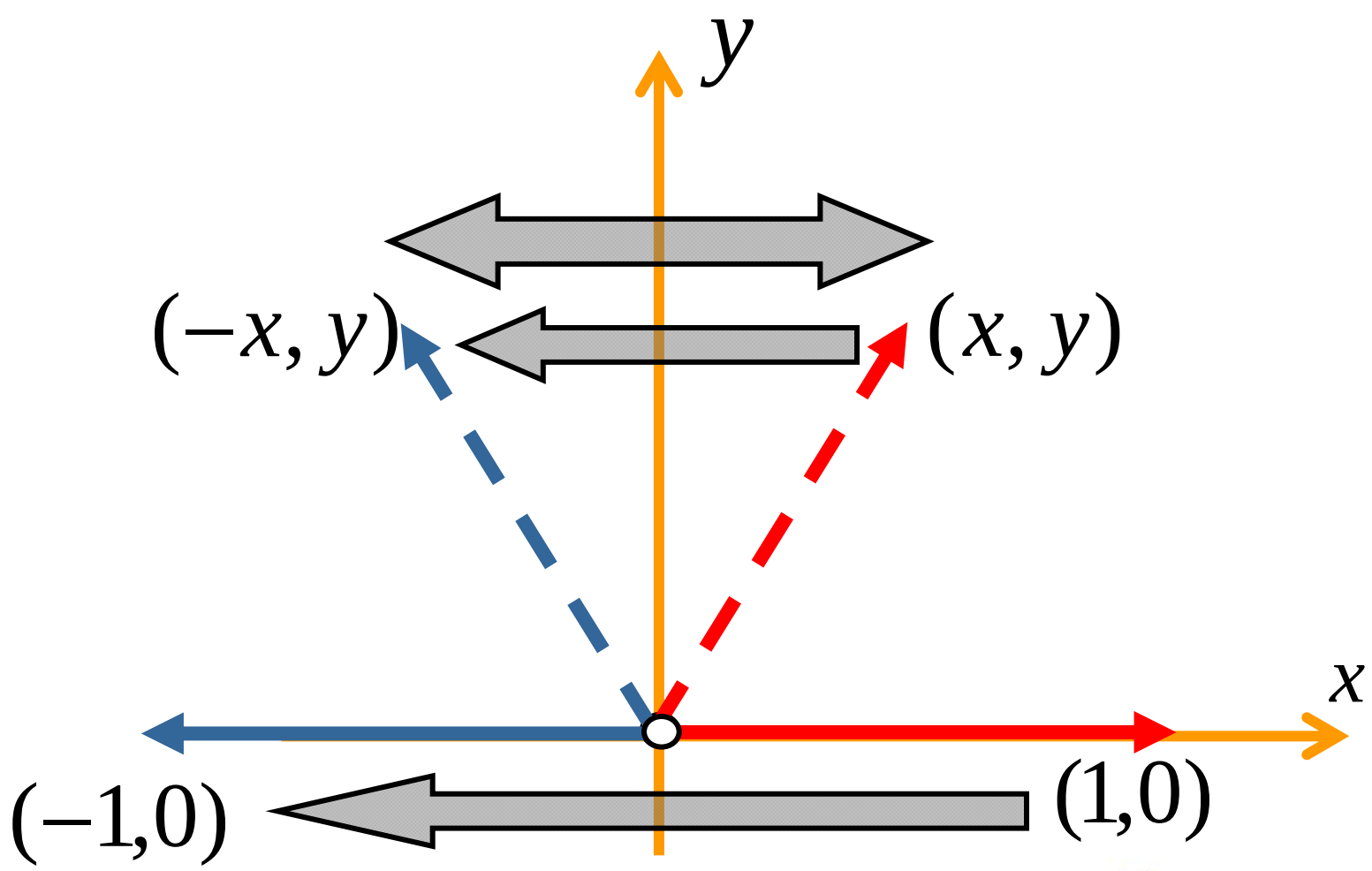
Refection *about* y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$x \leftrightarrow -x$$



Reflection *about* y-axis





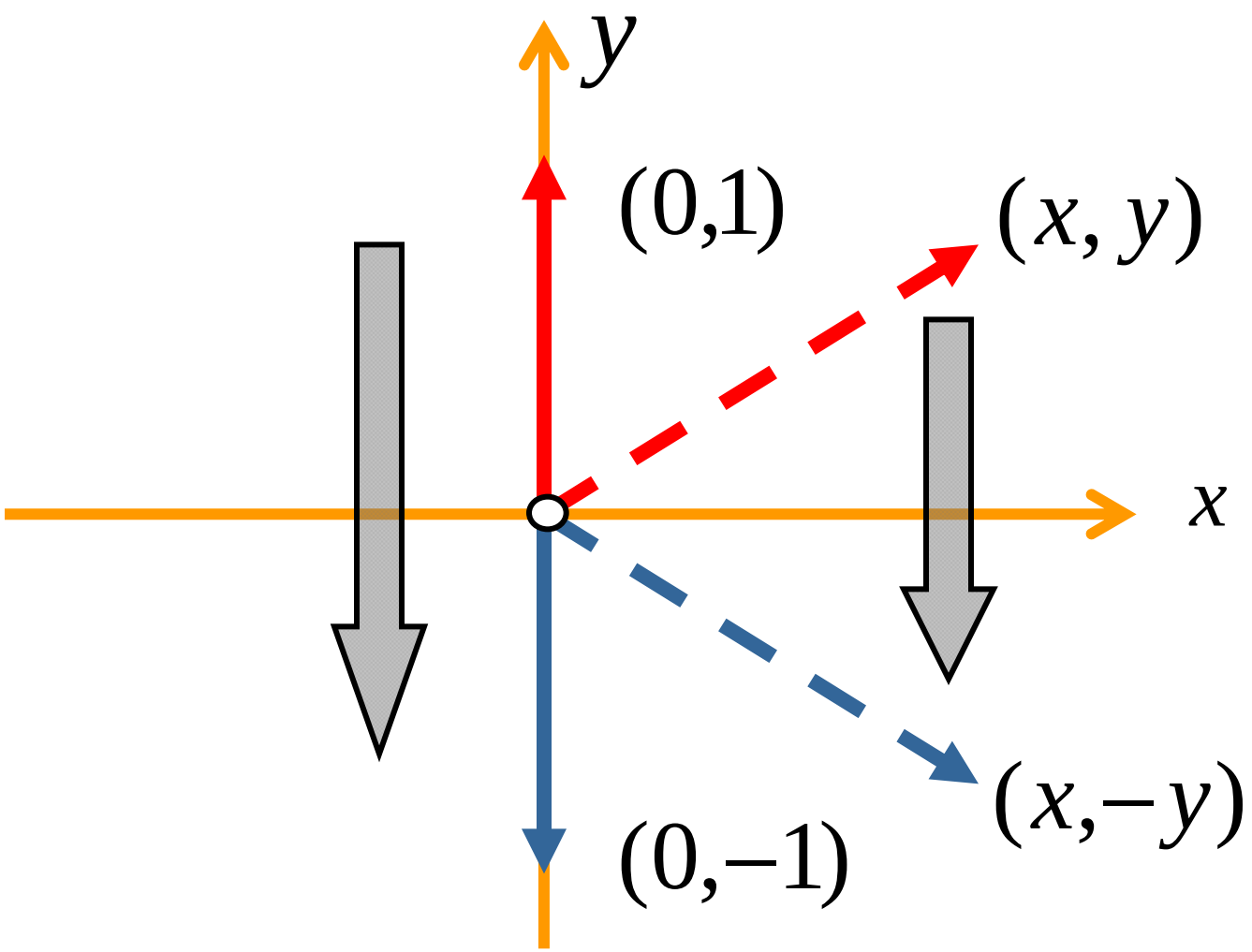
Reflection *about* x-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

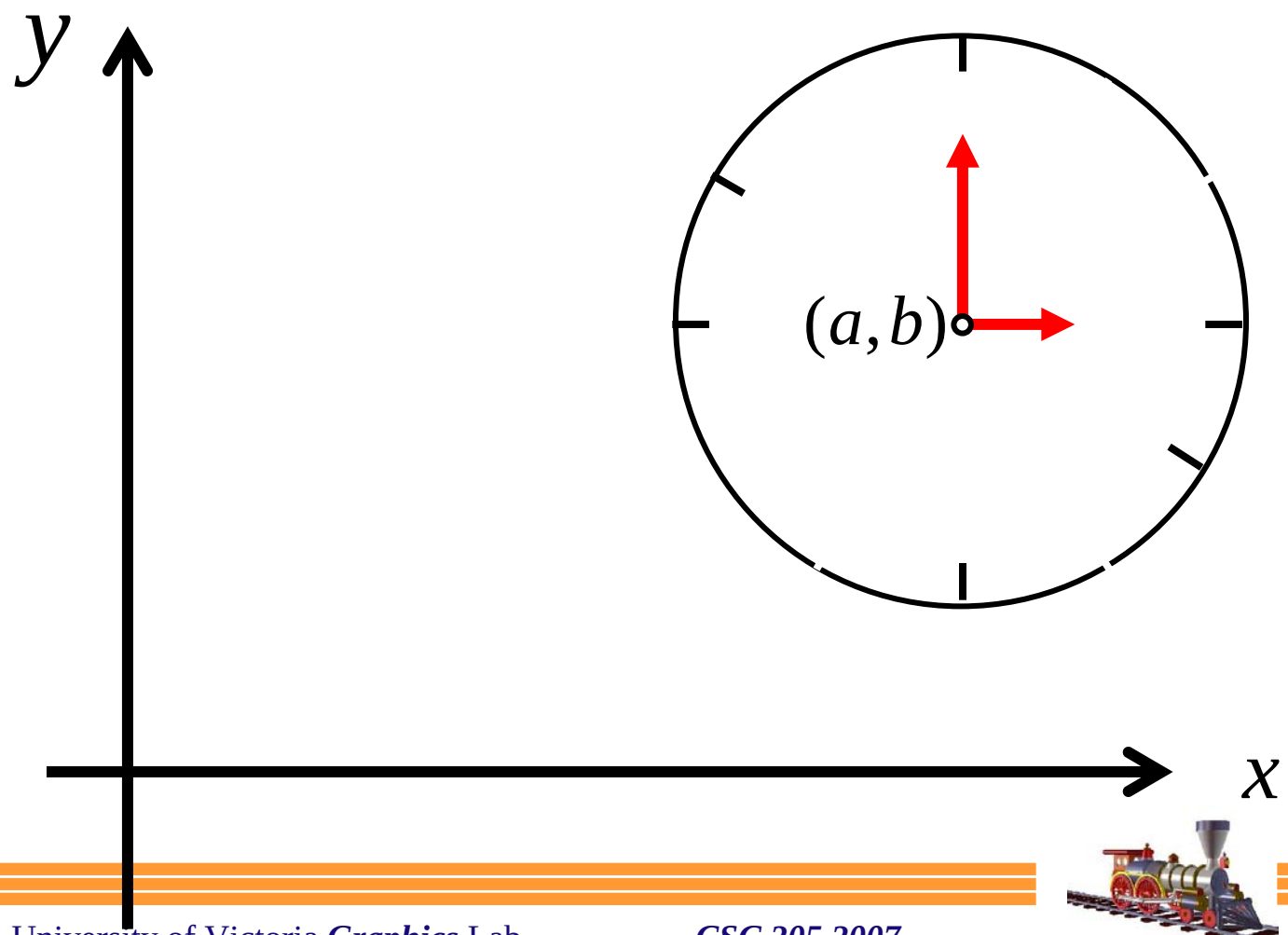
$$y \leftrightarrow -y$$



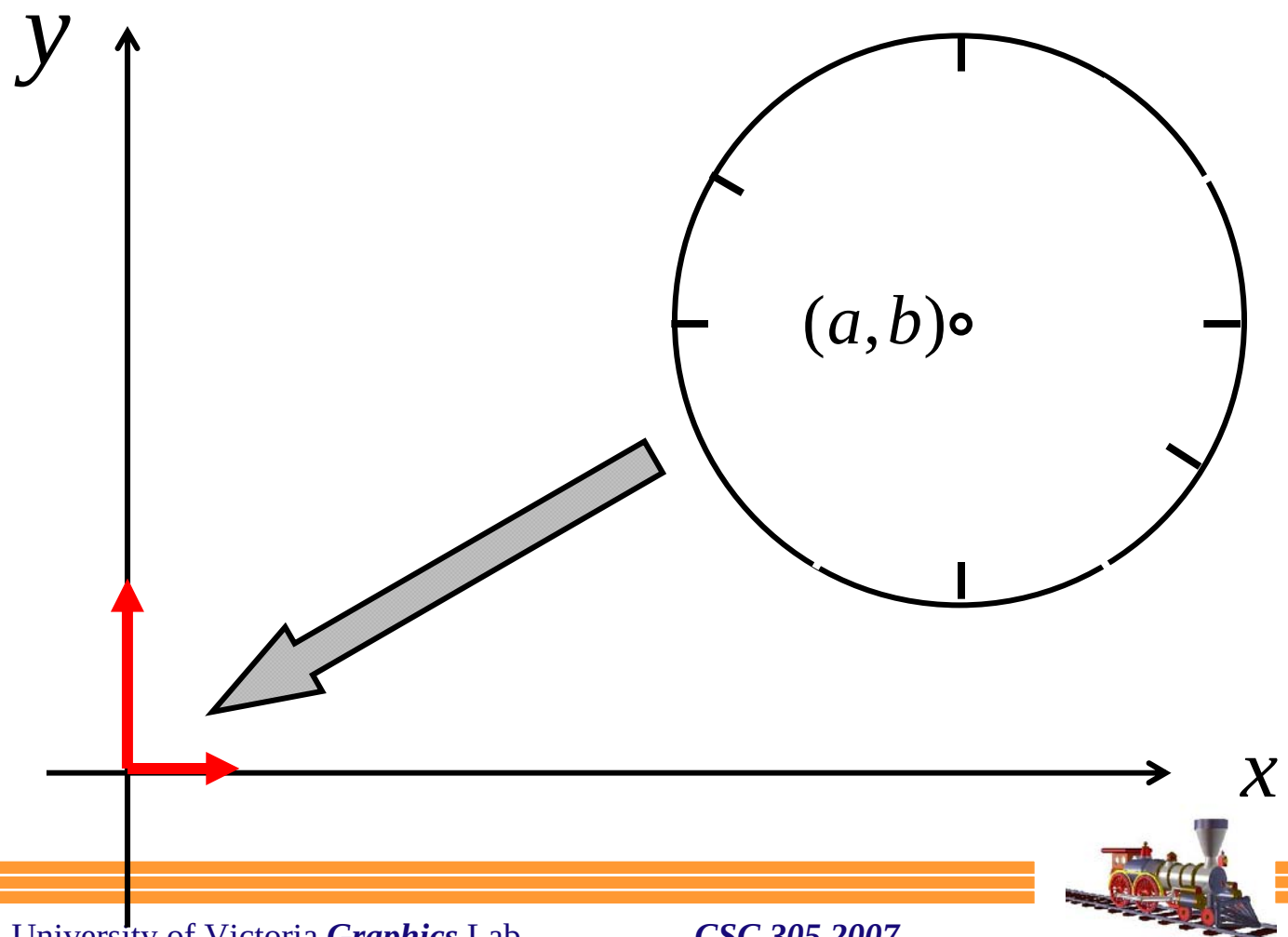
Reflection *about* x-axis



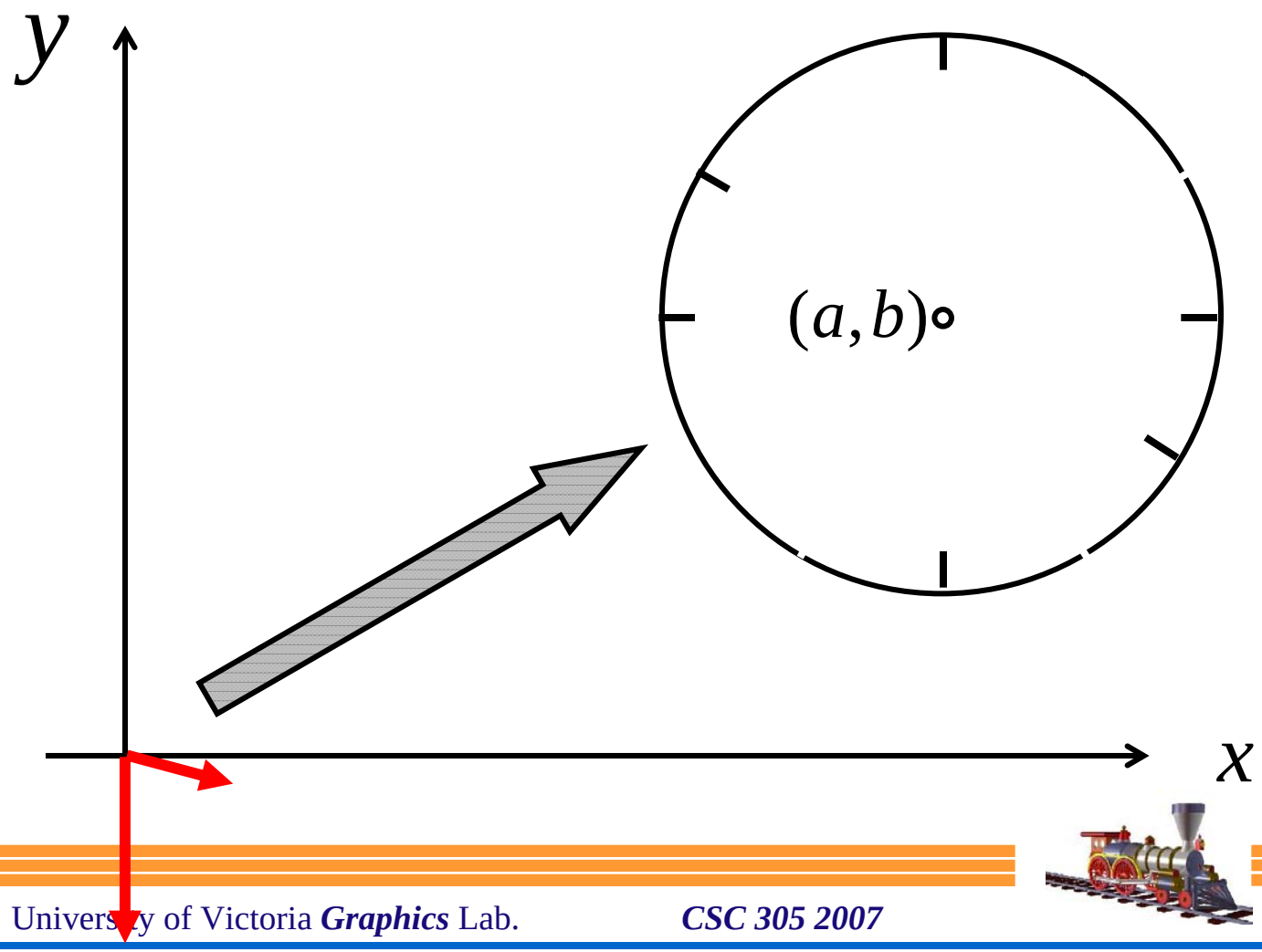
Ex: Advance clock hands



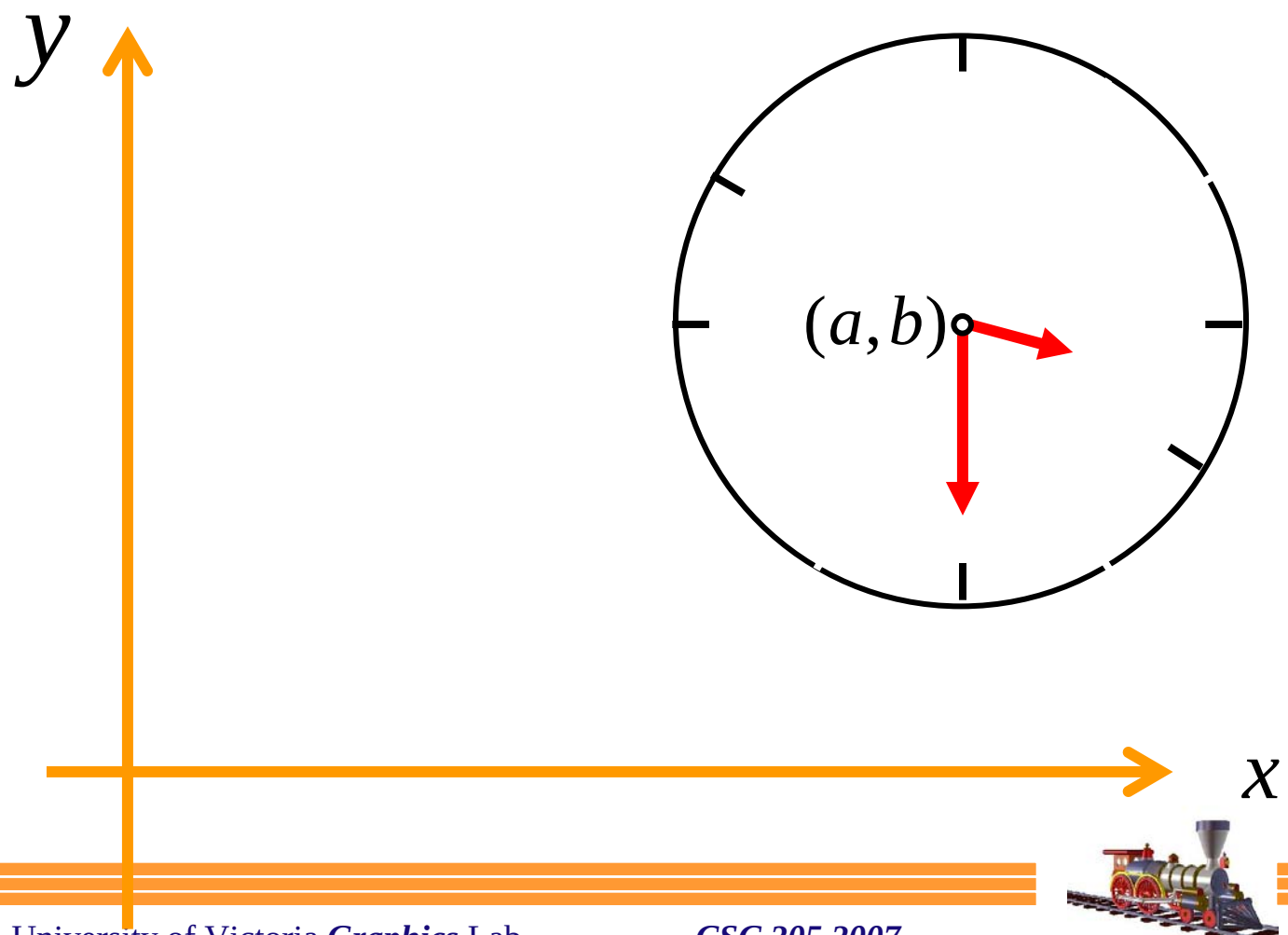
Ex: Advance clock hands



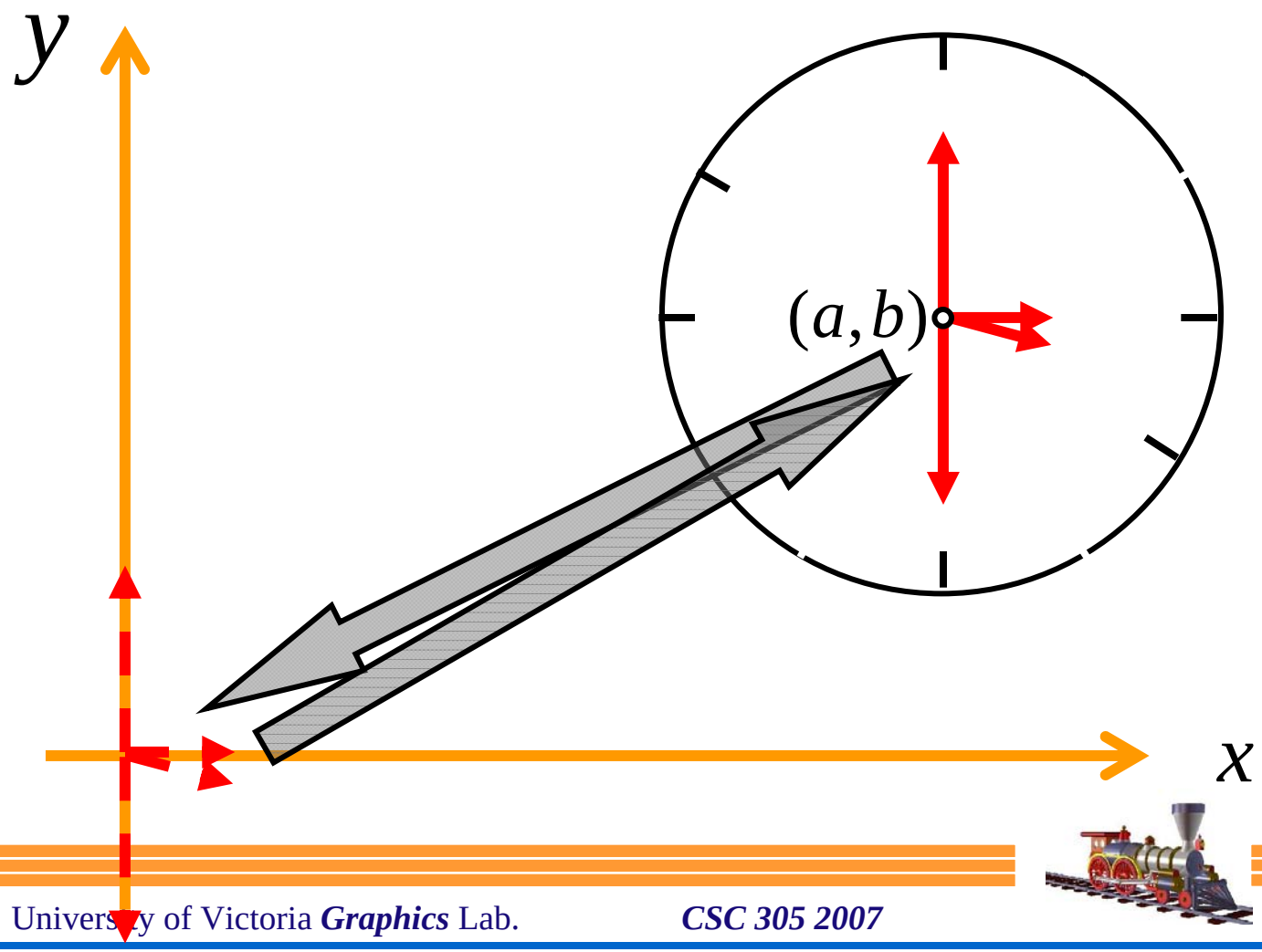
Ex: Advance clock hands



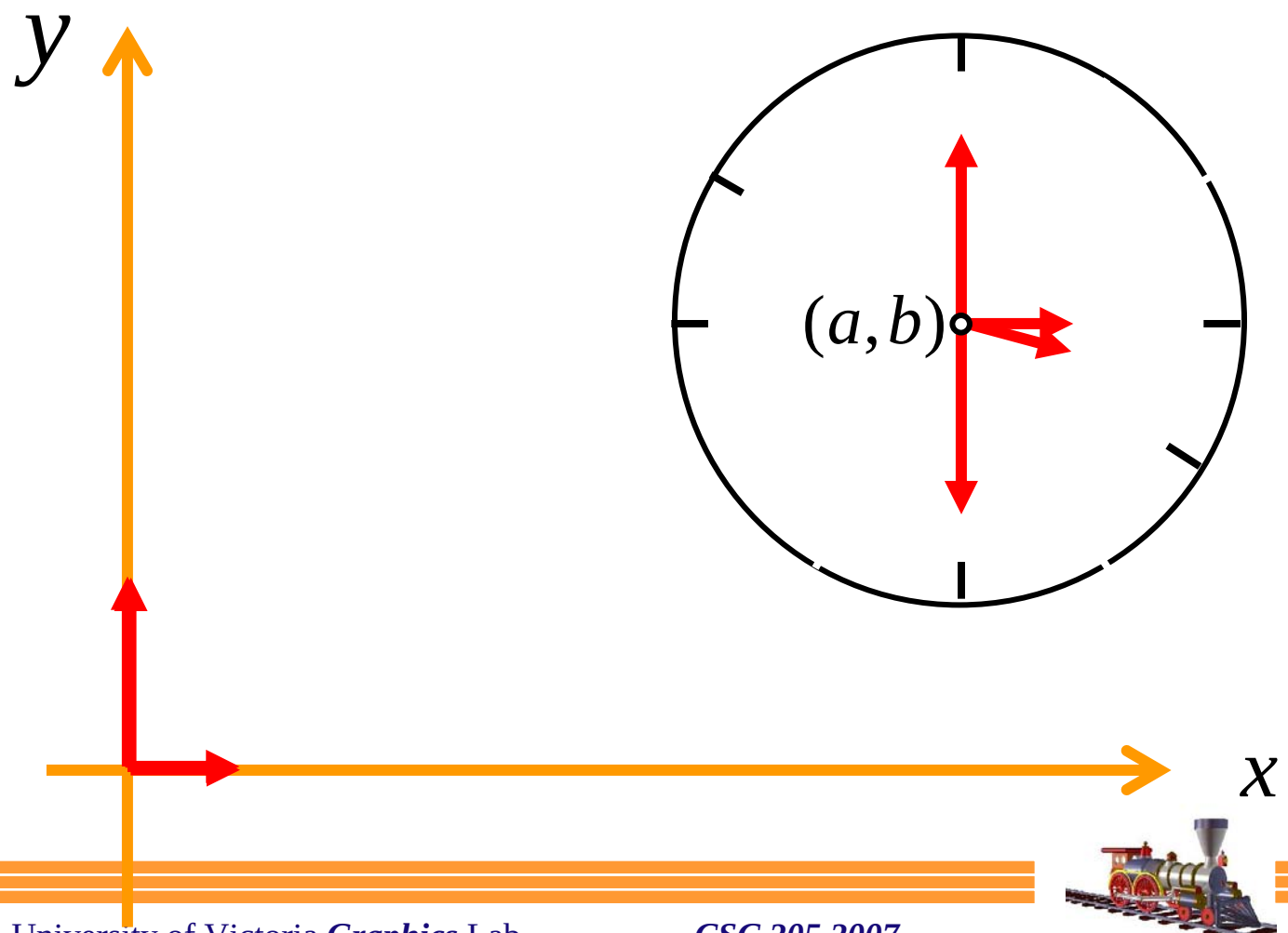
Ex: Advance clock hands



Ex: Advance clock hands



Ex: Advance clock hands



Clock

- ***Translate*** **to** ***Origin***
- ***Move*** **hand with** ***rotation***
- ***Move*** **hand back to clock**
- ***Do*** **other hand**



Clock Transformations

$$T_S = T(a,b)\ R(t)\ T(-a,-b)$$

$$T_b = T(a,b)\ R(12 * t)\ T(-a,-b)$$

$$where \quad t = -15^\circ$$



Clock Transformations

$$\underbrace{\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Translate Back}} \underbrace{\begin{bmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Rotate About Origin}} \underbrace{\begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Translate to Origin}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$



3D Transformations

- ***Scale*** $S_x(\lambda),\ S_y(\lambda),\ S_z(\lambda)$
- ***Rotate*** $R_x(\theta),\ R_y(\theta),\ R_z(\theta)$
- ***Translate*** $T_x(d), T_y(d), T_z(d)$
- ***Shear*** $Sh_x(d),\ Sh_y(d),\ Sh_z(d)$



3D Scale in x

$$S_x(\lambda) = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



3D Scale in x

$$S_x = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda x \\ y \\ z \\ 1 \end{bmatrix}$$



3D Scale in y

$$S_y(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ \lambda y \\ z \\ 1 \end{bmatrix}$$



3D Scale in z

$$S_z(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \lambda z \\ 1 \end{bmatrix}$$



Overall 3D Scale

$$S(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (1/\lambda) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ (1/\lambda) \end{bmatrix}$$



Overall 3D Scale

Same in *x*,*y* and *z*:

$$\begin{bmatrix} x \\ y \\ z \\ (1/\lambda) \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

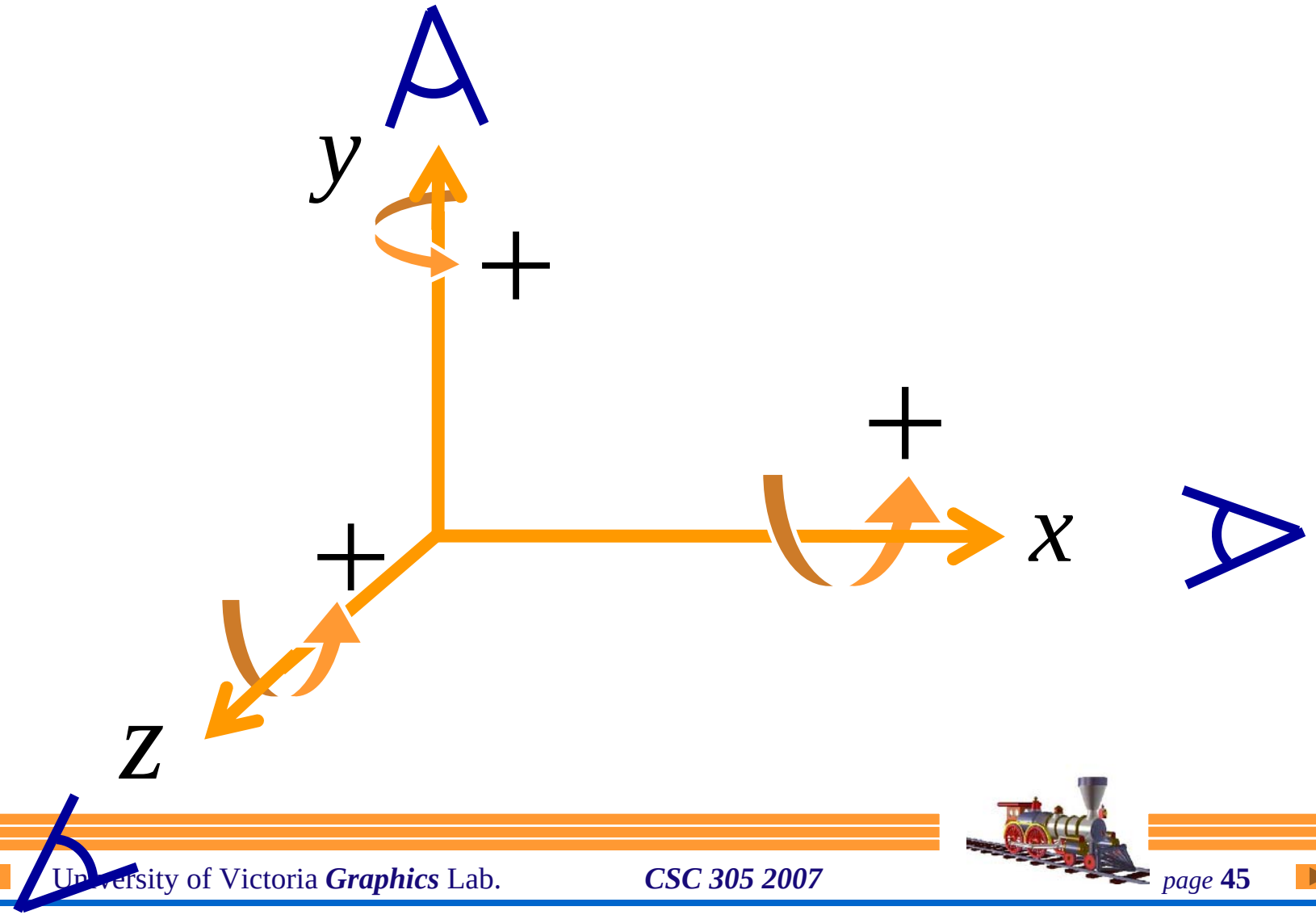


Positive Rotation *in* 3D *?*

- Sit at $+\infty$ end of given axis
- Look at Origin
- CC Rotation is in *Positive* direction



Positive Rotations



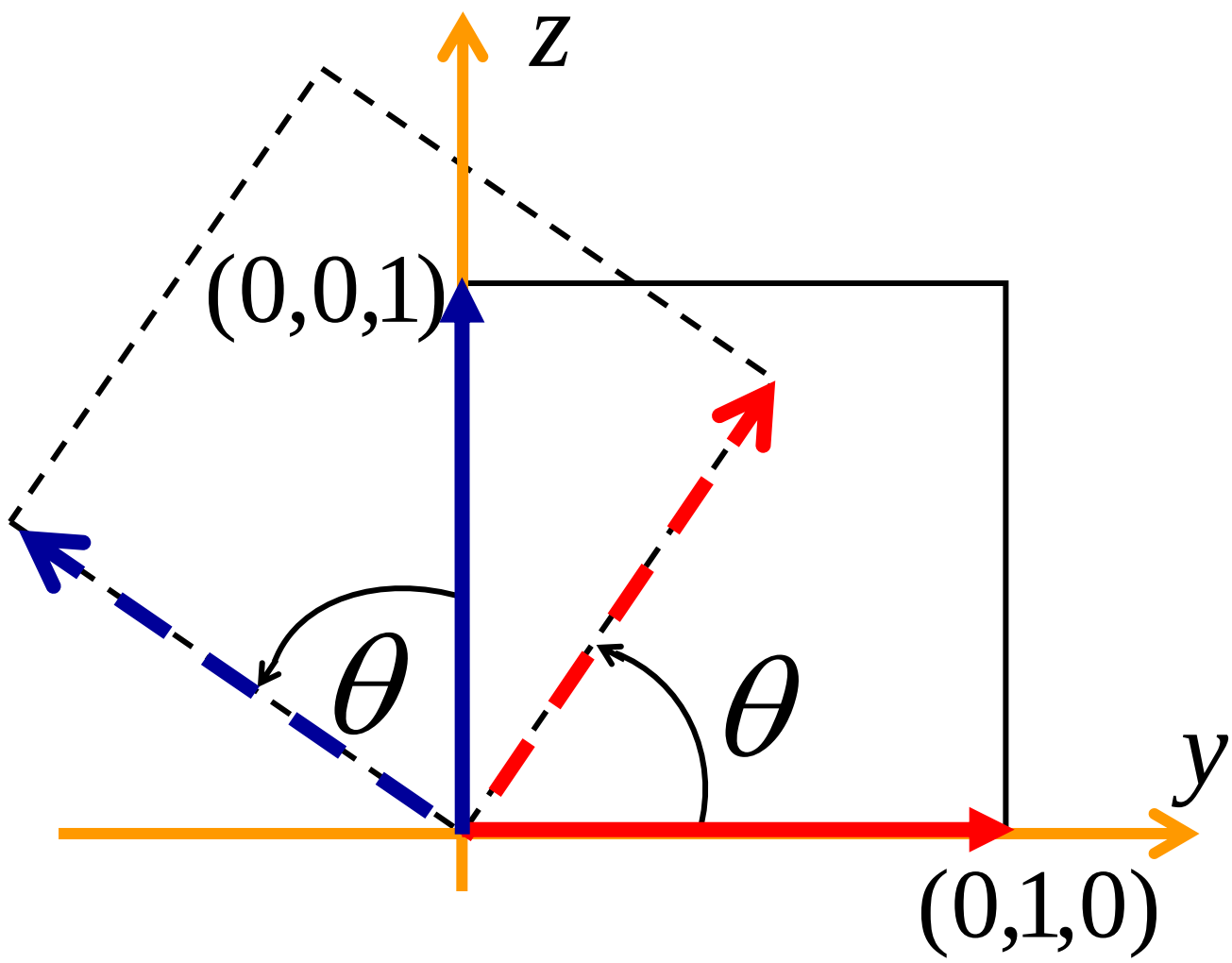
3D Rotation *about* z-axis *by* θ

We have already done this:

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



3D Rotation *about* x-axis *by* θ



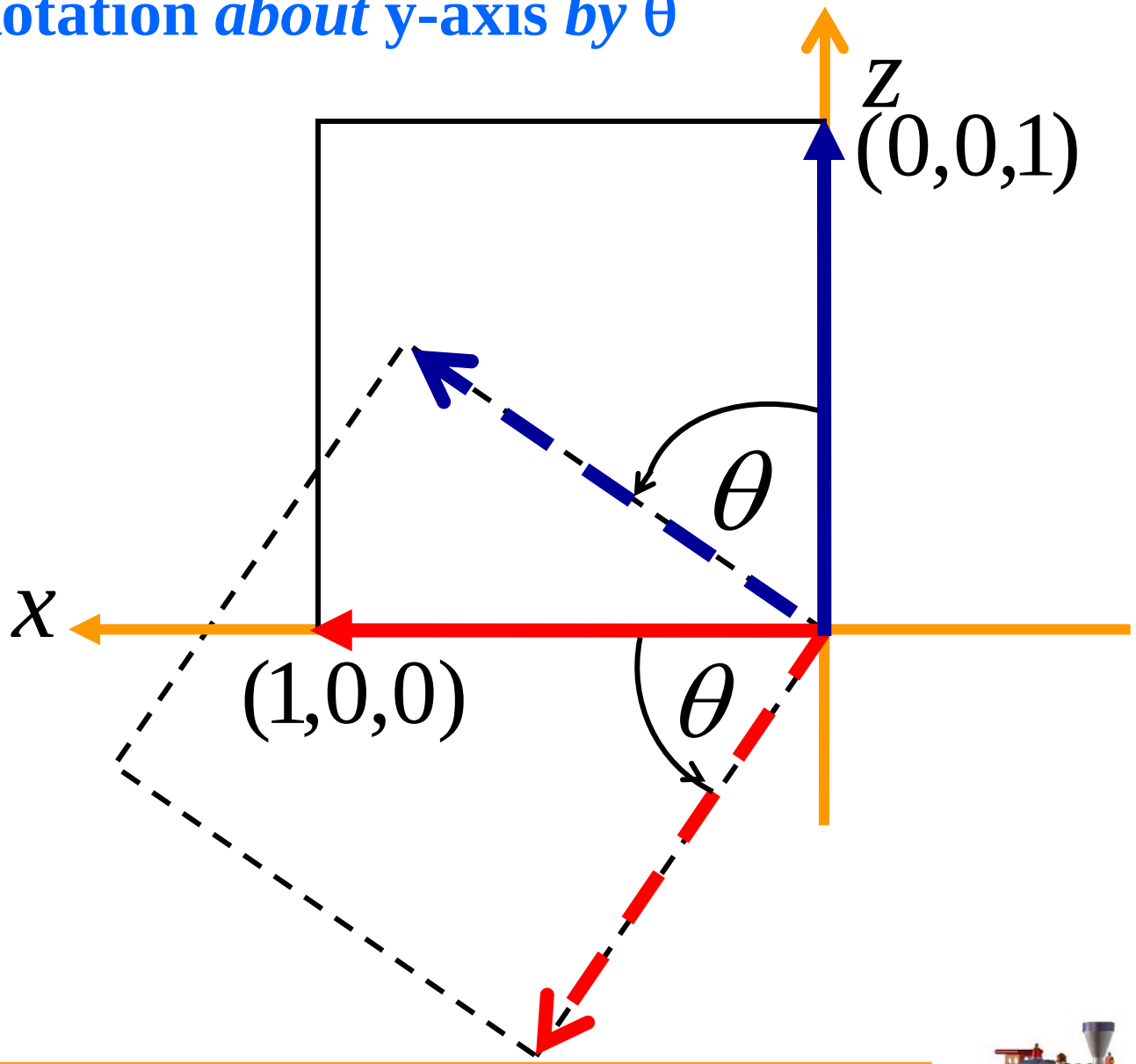


3D Rotation *about* x-axis *by* θ

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



3D Rotation *about* y-axis *by* θ



3D Rotation *about* y-axis *by* θ

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Elementary Transformations

- ***Scale*:** $S_{\lambda_x}(v),\ S_{\lambda_y}(v)$
- ***Rotate*:** $R_{\theta_x}(v),\ R_{\theta_y}(v)$
- ***Translate*:** $T_{d_x}(v),\ T_{d_y}(v)$
- ***Shear*:** $Sh_{\lambda_x}(v),\ Sh_{\lambda_y}(v)$
- ***Reflect*:** $Rf_x(v),\ Rf_y(v)$

