The Dominating Set Problem is Fixed Parameter Tractable for Graphs of Bounded Genus

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February 20, 2004

Abstract

We describe an algorithm for the dominating set problem with time complexity $O((4g+40)^k n^2)$ for graphs of bounded genus $g \geq 1$, where k is the size of the set. It has previously been shown that this problem is fixed parameter tractable for planar graphs. We give a simpler proof for the previous $O(8^k n^2)$ result for planar graphs. Our method is a refinement of the earlier techniques.

Keywords. graph, genus, dominating set, fixed parameter algorithm.

1 Introduction

The DOMINATING SET problem (DSP) is defined as follows.

Input: A graph $G = (V, E)$ and an integer parameter k

Question: Does there exist a dominating set of size k or less for G, i.e., a set $V' \subseteq V$ with $|V'| \leq k$ and such that for all $u \in V - V'$ there is a v in V' for which $uv \in E$?

This is a classic NP-complete problem [GJ79] which is also apparently not *fixed parameter tractable* (with respect to the parameter k) because it is known to be $W[2]$-complete in the W-hierarchy of fixed parameter complexity theory [DF99]. In this theory, any graph problem for which there is an algorithm with time complexity $O(f(k)n^\alpha)$, for some problem parameter

k, where n is the number of nodes in the graph and where α is a constant independent of k and n, is said to be *fixed parameter tractable* (fpt).

DSP is fixed parameter tractable with respect to, for example, treewidth [AP89] and tree decompositions are computable in linear time, for fixed treewidth [Bod96]. DSP is similar in definition to the vertex cover problem (VCP), but they seem to differ considerably in their fixed parameter tractability properties. The Robertson-Seymour theory of graph minors [RS90] can be used to show that VCP is fixed parameter tractable because vertex cover is closed with respect to taking minors, and fpt algorithms have been described [DF99] for VCP. But DSP is not closed with respect to taking minors.

DSP remains NP-complete when restricted to planar graphs [GJ79]. Fellows and Downey [DF95, DF99] gave a "search tree" algorithm for this problem which has time complexity $O(11^k n)$, when the input is restricted to planar graphs. The best search tree algorithm so far, which uses *kernelization*, is described in [AFF$^+$01] and has a time complexity of $O(8^k n)$. A tree decomposition based algorithm with time complexity $O(c^{\sqrt{k}} n)$, where $c = 4^{6\sqrt{34}}$, is given in [ABFN00]. Recently, in [AFN02], it has been shown that the problem can be reduced to a kernel of size linear in k.

In this paper, using the search tree approach, we show that the dominating set problem is fixed parameter tractable for graphs of bounded genus. For graphs of genus $g \geq 1$ we prove a time complexity of $O((4g+40)^k n^2)$. The graph genus problem is itself NP-complete [Tho89], but, for fixed genus, there is a linear time algorithm [Moh96].

In Section 2 we review the basic search tree algorithm previously used by others, and we define the reduction process whereby it is ensured that some node of restricted degree will exist after the reduction is applied. In Section 3 we set up some lemmas to be used in both of the following two sections. In Section 4 we present a shorter, simpler proof of the existing result that DSP is fixed parameter tractable for planar graphs, without improving on the time complexity of the earlier result. In Section 5 we extend the results to graphs of bounded genus. These latter results were previously presented in [EFF02].

2 The Algorithms

2.1 The Basic Algorithm

The result in [AFF$^+$01] is based on the following simple algorithm, Figure 1. We can extend the concept of domination from dominating all the nodes V of a graph to dominating some subset of V. The node set B is a subset of the nodes of G which are to be dominated. E is an edge set and *exists* is a global Boolean variable. $N(v)$ denotes the neighbors of v (not including v itself) with respect to the edge set E and $I(v)$ denotes the set of edges incident with v.

The claim that the global variable *exists* is set to true iff there exists a dominating set for the nodes in B of cardinality not greater than k follows from the observation that for every node u in B either u or one of its neighbors is in the dominating set. Hence we can

procedure dominating_set (B : node_set, E : edge_set, k : integer);

{The global variable *exists* is set to true iff there exists a dominating set for the nodes in B of cardinality not greater than k}

> **if** $|B| \leq k$ **then** exists $\leftarrow$ true
> **else if** $k > 0$
>   **then** Choose some $u \in B$;
>   **for** all $v \in (N(u) \bigcup \{u\})$ **do**
>   dominating_set $(B - (N(v) \bigcup \{v\}), E - I(v), k - 1)$;

Figure 1: The Basic Algorithm

decide the dominating set question for any graph $G = (V, E)$ and integer k by setting *exists* to false and invoking dominating_set(V, E, k).

Suppose the maximum degree of any node in B is d and let $t(m)$ denote the time complexity of the procedure, excluding the recursive invocation, where $m = |E|$. Then the time complexity $T(m)$ of the procedure is described by the recurrence relation: $T(m, k) \leq t(m) + (d+1)T(m-1, k-1)$. The solution to this recurrence is $T(m, k) = O((d+1)^k t(m))$, as can be seen by substituting $T(m, k) \leq (c(d+1)^k - b)t(m)$, for some constants c and b, in the recurrence and assuming that $d \geq 2$ as we will see is the case.

It is straightforward to adjust node and edge sets and even to copy entire data structures for the recursive call, all in time linear in the number of edges. Noting that for graphs of bounded genus the number of edges is $O(n)$, where n is the number of nodes, we see that this simple algorithm has time complexity $O((d+1)^k n)$ and the dominating set problem for any family of graphs with bounded degree is fixed parameter tractable. Of course in general there is no bound on the degree of the input graph.

2.2 The Reduction Procedure

To make use of this algorithm for graphs of bounded genus, but not of bounded degree, a reduction procedure is applied to the graph at each level of recursion. This procedure does not change the minimum size of a dominating set but guarantees that there is at least one node in B of bounded degree. Figure 2 defines the reduction procedure. Rules R1 through R6 are among the reduction rules used in [AFF$^+$01]. R7 is a new rule which generalizes one of the old rules. Given this procedure, we replace the statement "Choose some $u \in B$" in the basic algorithm by two statements: "Invoke Reduce; Choose some $u \in B$ of minimum degree".

Let B' and E' be the node and edge sets resulting from the application of any one of the rules in the reduction procedure to the sets B and E. One can examine each of the rules to see that there exists a dominating set for the nodes in B with respect to the edges in E of

procedure Reduce;
{We assume the node sets B and E as in Figure 1.
For convenience, we refer to the nodes in B as the "black" nodes and the rest, $V - B$, as the "white" nodes.}

repeat

R1: Remove all edges between white nodes from E;

R2: Remove all edges incident with degree 1 white nodes from E;

R3: If there is a black node w of degree 1, incident with a node u,
then remove all neighbors of u from B; add u to B;
remove all edges incident with u from E;

R4: If there is a white node u of degree 2, with black neighbors u_1 and u_2,
and an edge $u_1u_2 \in E$, then remove the edges uu_1 and uu_2 from E;

R5: If there is a white node u of degree 2, with black neighbors u_1, u_3,
and there is a black node u_2 and edges u_1u_2 and $u_2u_3 \in E$,
then remove the edges uu_1 and uu_3 from E;

R6: If there is a white node u of degree 3, with black
neighbors u_1, u_2, u_3 and there are edges u_1u_2 and $u_2u_3 \in E$
then remove the edges uu_1, uu_2 and uu_3 from E;

R7: If there are white nodes u_1 and u_2 such that $d(u_2) \leq 7$ and $N(u_1) \subseteq N(u_2)$,
then remove all the edges $\in E$ incident with u_1 from E;

until no change in E;

Figure 2: The Reduction Procedure

cardinality k iff there exists a dominating set for the nodes in B' with respect to the edges in E' of cardinality k.

A graph whose node set is partitioned into black and white nodes will be referred to as a *black and white* graph. The black nodes are those yet to be dominated. The white nodes are those already dominated. A graph that is derived by applying the reduction procedure to a black and white graph will be referred to as a *reduced* black and white graph.

2.3 Time Complexity of the Reduction Process

We justify an upper bound of $O(n^2)$ on the the time complexity, $t(n)$, of the reduction procedure. We assume the use of two data structures representing the graph and associated information:

- An elaborated adjacency list in which each list is a two way linked list, and for all edges $\{u, v\}$ the entry for u in the list for v is linked directly to the entry for v in the list for u, and *vice versa*. With these extra links, an edge, once identified can be removed in constant time. A black/white indicator and a degree count are associated with each node.

- A standard adjacency matrix permits the existence of an edge to be tested in constant time, and likewise the removal of an edge.

With these data structures in mind, consider the work done in executing an entire sequence of k invocations of the dominating set procedure. It can be seen that for all rules except R7, $O(n)$ is an upper bound on the work involved in each rule. For rule R7, it seems that, although no work is required unless the degree of a white node is reduced to seven or less, any such node must be compared to all other white nodes. Thus we see no better bound than $O(n^2)$.

3 Some Useful Lemmas

We set up some useful lemmas that will be used in both of the two following sections. Let $\mathcal{S}_g$ be the orientable compact surface of genus g. An embedding of a graph G in $\mathcal{S}_g$ is a drawing in $\mathcal{S}_g$ such that each node corresponds to a point and each edge to a line, i.e., a simple curve with two ends, and such that two lines corresponding to a pair of edges are internally disjoint, i.e., do not intersect or intersect only at their ends. The genus of G, written $\gamma(G)$, is the minimum g such that G can be embedded in $\mathcal{S}_g$. By an embedded graph we mean a graph together with its embedding in a surface, which allows us to talk about both combinatorial and geometrical properties. For instance, an edge of an embedded graph can also be seen as a line connecting two end points.

Let G be an embedded graph. A face, F, of G is a connected component of $\mathcal{S}_g - G$. The boundary of F corresponds to a group of cyclic sequences of edges bounding the face. The degree of a face F, written $d_G(F)$ or $d(F)$, is the number of edges appearing in the boundary, edges appearing twice being counted twice. The boundary-graph of F, written $b(F)$, is the subgraph of G consisting of those nodes and edges of G incident with F. A face is said to be a 2-cell if it is homeomorphic to an open disk. G is a 2-cell embedding if every face is a 2-cell. The boundary-graph of F could contain a cut edge, i.e., an edge whose removal disconnects the graph. A cut edge can appear twice in a boundary. A face of degree d is called a d-face. If $b(F)$ is a cycle in G, then F is called a proper face.

Lemma 3.1 [Tho95, Chapter 5, Theorem 3.8] *Every embedding of G in $S_{\gamma(G)}$ is a 2-cell embedding.*

Lemma 3.2 *Let $G = (V, E)$ be a connected graph. If G has a 2-cell embedding in $\mathcal{S}_g$ $(g \geq 0)$ such that there are no 1-faces and no 2-faces, then*

$$|E| \leq 6(g-1) + 3|V| \tag{1}$$

Proof
The following relation is proved in [Tho92]: $n - e + f \geq 2 - 2g$, where n is the number of nodes, e is the number of edges and f is the number of faces in the embedding. Because there are no 1-faces or 2-faces, $3f \leq 2e$. Hence $n - e + 2e/3 \geq 2 - 2g$, and (1) follows. □

Let $G = (V, E)$ be a graph and x, y be nodes of G, we denote by $G + xy$, $G - xy$ the simple graphs $(V, E \cup \{xy\})$ $(V, E \backslash \{xy\})$. We denote by $\delta(G)$ the minimum degree of G.

Lemma 3.3 *Let $G = (V, E)$ be a connected simple graph for which $\delta(G) \geq 3$. If G is not 3-connected then G has a 2-node cut $\{x, y\}$ such that there is a component C of $G - \{x, y\}$ such that $G[V(C) \cup \{x, y\}] + xy$ is 3-connected.*

Proof
Because $\delta(G) \geq 3$, $|V(G)| = 4$ implies that G is isomorphic to K_4, which is 3-connected. No smaller graph is 3-connected.

Suppose $|V(G)| > 4$ and G is not 3-connected. Then either there exists a cut node $\{x\}$ which is in a pendant block or, if not, there exists a 2-cut $\{x, y\}$.

There is no cut node

Then there exists a 2-cut $\{x, y\}$. Let C be a component of $G - \{x, y\}$ and let $G' = G[V(C) \cup \{x, y\}] + xy$. G' is 2-connected. Any such C has at least 2 nodes, else there is a node in G of degree 2. Hence all such G' contain at least 4 nodes.

If $|V(G)'| = 4$ then $d_{G'}(x) = d_{G'}(y) = 3$, else there is a node in G of degree 2. Hence G' is isomorphic to K_4.

If $|V(G')| > 4$, consider the following procedure.

while G' is not 3-connected **do**
  Let $\{a, b\}$ be a 2-cut in G';
  Let C be a component in $G' - \{a, b\}$ containing neither x nor y;

$x \leftarrow a$; $y \leftarrow b$;
$G' \leftarrow G[V(C) \cup \{x, y\}] + xy$
endwhile

G' does not contain a cut node, hence, if it is not 3-connected there is a 2-cut $\{a, b\}$ in G' such that $\{a, b\} \neq \{x, y\}$. We note that $\{a, b\}$ is also a cut in G. The procedure must terminate, with G' being 3-connected, because the number of nodes in G decreases at each iteration. If not before, it must terminate when $|V(G')| = 4$ because then $G' = K_4$, else one of the remaining two nodes had degree less than three in G.

There exists a cut node

There must exist a cut node x which is in some pendant block (a maximal 2-connected component of G) say B. Note that since $\delta(G) \geq 3$, B must contain at least 4 nodes.

If B is 3-connected, then $\{x, a\}$, where a is a neighbour of x in B, is a 2-cut in G satisfying the lemma.

If B is not 3-connected, then $|V(B)| > 4$ and there exists a 2-cut, say $\{u, v\}$, in B, since there are no cut nodes. One of the components, say C, in $B - \{u, v\}$ does not contain x. If $B[C \bigcup \{u, v\}] + xy$ is not three connected then, by the argument used in the 2-connected case, there exists another 2-cut in C with the desired properties. □

Lemma 3.4 *If $G = (B \bigcup W, E)$ is reduced, then the white nodes constitute an independent set.*

Proof
This follows from rule R1 of the reduction procedure, which is inside the loop. Hence, although rule R3 can add white nodes and hence maybe edges between white nodes, rule R1 will be again enacted and remove any edges between white nodes. □

Lemma 3.5 *For all d, if there exists a reduced black and white graph G in which the minimum degree of any black node is d then there exists a reduced graph G' in which the minimum degree of any black node is also d and no white node has degree less than 3.*

Proof
By rule R2 of the reduction procedure, G has no white nodes of degree 1. Let u be a white node of degree 2 with neighbours x and y, which are necessarily black, by rule R1. By rule R4, there is no edge xy. Let G' be obtained from G by replacing all white nodes of degree 2 by an edge between their neighbours. Note that this operation does not change the the minimum degree of any black node. Suppose G' is not reduced. Then the only rule that could possibly apply is R6, i.e., there is some white node u of degree 3 in G' with black neighbours and edges between two pairs of these neighbours. Because G is reduced, rule R6 did not apply to u in G. Hence one of the edges between the neighbours of u must have been created in the derivation of G' from G, i.e., one of these edges replaced a white node

u' of degree 2. But this implies that R7 would apply in G to u and u', contradicting the assumption that G is reduced. $\square$

4 The Planar Case

In this section we show that a reduced, black and white, *planar* graph contains a black node of degree at most 7. This result was previously proved in [AFF+01]. Here we give a significantly shorter and simpler proof.

A block in a graph G is a maximal 2-connected subgraph of G. A pendant block is a block containing exactly one node that is a cut node in G. By a *plane graph* we mean an embedded graph in a plane, $\mathcal{S}_0$. It is known that a 2-connected plane graph must be a 2-cell embedding in which every face is proper. We say that a reduced, planar, black and white graph G is *full* if the addition of any edge to G results in a graph that is either not planar or not reduced.

Lemma 4.1 *Let $G = (B \bigcup W, E)$ be a connected, full, reduced, plane, black and white graph of minimum degree at least three. Then:*

1. *G is 2-connected.*
2. *G has only 3-faces and 4-faces.*
3. *The boundary of every 4-face contains exactly two white nodes.*
4. *Every white node of degree 3 must be in the boundaries of at least two 4-faces.*
5. *If $\{x, y\}$ is a cut of G, then $\{x, y\} \subset B$.*

Proof

1. Suppose, contrary to the claim, that G is not 2-connected. Then G has a face F that is not proper, i.e., $b(F)$ is not a cycle of G. Since G is a plane graph, $b(F)$ is connected, it contains a cut node and each block of $b(F)$ is either an edge or a cycle. Further, $b(F)$ contains at least two pendant blocks and each such pendant block is a cycle because $\delta(G) \geq 3$. Let C_1 and C_2 be two pendant blocks of $b(F)$ containing cut nodes a_1 and a_2 respectively. Note that a_1 and a_2 may be identical.

   Since W is an independent set, by Lemma 3.4, and $|V(C_i)| \geq 3$, $C_i - a_i$ must contain a black node. Let b_i be any black node in $V(C_i - a_i)$, for $i = 1, 2$. If $b_1 b_2 \in E$ then C_1 and C_2 are not pendant blocks of G and so it must be that $b_1 b_2 \notin E$.

   Consider $G + b_1 b_2$. Since we can draw the line $b_1 b_2$ in F, $G + b_1 b_2$ is a planar graph. By inspection of the reduction rules we see that, since G is reduced, none of the reduction rules can apply to $G + b_1 b_2$ unless b_1, b_2 have a white, degree three neighbor in common, in which case R6 might apply. So $G + b_1b_2$ is a reduced graph, except perhaps for this possibility.

If b_1, b_2 do have a white neighbor in common, then it must be that $a_1 (= a_2)$ is that neighbor. But $d_{C_1}(a_1) = d_{C_2}(a_1) = 2$ and so $d_G(a_1) \geq 4$, and R6 can not apply. Hence $G + b_1b_2$ is a reduced, planar, black and white graph, contradicting the assumption that G is full.

2. By (1) G is 2-connected, so every embedding of G is a 2-cell embedding and the boundary of each face is a cycle of G. G has no 1-face or 2-face since G is a simple graph. We now show that no boundary of any face of G can contain more than three black nodes. Suppose, on the contrary, that there exists a face F containing four black nodes b_1, b_2, b_3, b_4 in its boundary $b(F)$, appearing in that order.

   - If $b_1b_3 \notin E$ then b_1, b_3 do not share a white neighbor because of reduction rule R5. Hence $G + b_1b_3$ is a reduced, planar, black and white graph, because b_1b_3 can be added in F and the addition can not activate any reduction rules.
   - If $b_1b_3 \in E$ it must be drawn outside of F and hence b_2b_4 can not also be drawn outside of F and so $b_2b_4 \notin E$. Then, just as in the immediately previous item, it must be that $G + b_2b_4$ is a reduced, planar, black and white graph.

   So in both cases it is possible to add an edge without destroying planarity or activating any reduction rules, contradicting the assumption that G is full.

   Suppose there exists a face F such that $b(F)$ contains at least five nodes. Since the boundary can not contain more than three black nodes and the white nodes are independent, it can contain no more than six nodes. Thus the boundary is either of the form a_1, b_1, a_2, b_2, b_3 or $a_1, b_1, a_2, b_2, a_3, b_3$, where the a_i are white and the b_i black. In both cases it can not be the case that both edges a_1b_2 and a_2b_3 exist because they can not both be drawn outside of the face without intersecting. Whichever is not present can be added in F without destroying planarity. Further, by inspection of the reduction rules, the addition of an edge between a black and white node can not activate any rule. So it is possible to add an edge, contradicting the assumption that G is full. So all faces are either 3-faces or 4-faces.

3. We have just argued that the boundary of a face can not contain more than three black nodes. Hence, if F is a 4-face its boundary must contain white nodes. Since the white nodes are independent, $b(F)$ can not contain more than two white nodes. Suppose $b(F)$ contains only one white node a. Let $b(F) = ab_1b_2b_3a$, where the a is white and the b_i black.

   - If $ab_2 \notin E$, then ab_2 can be added to the face without destroying planarity. Since we assume the white node is already of degree at least three, the addition of the edge can not activate any reduction rule.

- If $ab_2 \in E$, it must be that $d_G(a) \geq 4$, else reduction rule R6 would already have been activated. It can not be that b_1b_3 exists because it can not be drawn outside of F without intersecting ab_2. Nor can b_1, b_3 share a neighbor for the same reason. Hence b_1, b_3 can be added in F without destroying planarity and without activating any reduction rules.

In both cases we contradict the assumption that G is full, so it can not be the case that $b(F)$ contains exactly one white node.

4. Suppose a is a white node with three neighbors b_1, b_2, b_3. At least two pairs of b_1, b_2, b_3 are not adjacent, else reduction rule R6 would already have been activated. Suppose b_i and b_j are not adjacent. Since all faces are either 3-faces or 4-faces by (2), a, b_i, b_j must be in the boundary of a 4-face, for at least two pairs i, j. Therefore a must be in the boundaries of two 4-faces.

5. Suppose, contrary to the claim, that $\{x, y\}$ is a cut and yet one or both of the nodes is white.

   - If $xy \in E$, then $G - xy$ is 2-connected and $\{x, y\}$ is a cut of $G - xy$, so that x, y must be in the boundaries of two faces, say F_1 and F_2 of $G - xy$, and it must be that $d_{G-xy}(F_1) \geq 4$ and $d_{G-xy}(F_2) \geq 4$. One of F_1, F_2, say F_1, must also be a face of G and it must be a 4-face by (2), since it can not be a 3-face. By (3) both x and y must be white. But then, by lemma 3.4, $xy \notin E$, contradicting our initial assumption.

   - Suppose $xy \notin E$. By the same argument as used in the first case, both x and y must be in the boundaries of two faces F_1 and F_2 of G and it must be that $d_G(F_1) = 4$ and $d_G(F_2) = 4$. By assumption in this case, x, y are not adjacent. Since at least one of $\{x, y\}$ is white, they must both be white, by (3).

     Suppose F_1 and F_2 share an edge, say xv. Since G is simple, without parallel edges, it must then be that they also share the edge uy and hence that u is of degree two, contradicting our original assumptions about G. So F_1 and F_2 do not share an edge and hence x and y are each incident to two edges in each face. Hence $d_G(x) \geq 4$ and $d_G(y) \geq 4$.

     There are two possibilities: either the two faces are nested one inside the other, in which case one is the outer, unbounded face of the embedding, or the faces are parallel to each other. See Figure 3.

     Suppose the faces are nested and let $b(F_1)$ be the outer, unbounded face. Let b_1, b_2 be the two black nodes in (F_1). By definition, b_1b_2 is not in the outer face nor can it be drawn elsewhere without intersecting other edges. Hence $b_1b_2 \notin E$. The parallel case is similar. So b_1b_2 can be drawn in F_1 without destroying planarity. Further, in both cases, b_1, b_2 can not share a neighbor except for one of $\{x, y\}$, which are of degree at least 4. So the edge b_1b_2 can be added to G without acti-

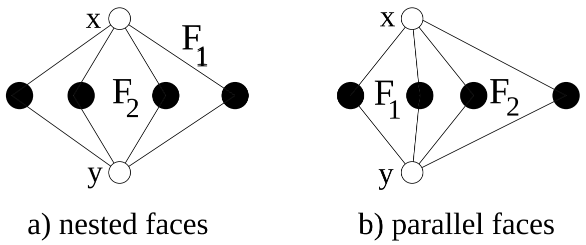

Figure 3: The relations between faces

vating any reduction rule. So G is not full, contradicting our initial assumption. □

Theorem 4.1 *If $G = (B \bigcup W, E)$ is a reduced, planar, black and white graph, then there exists a node $b \in B$ such that $d_G(b) \leq 7$.*

Proof
Suppose, contrary to the claim, that there exists a reduced, planar, black and white graph G_0 in which the degree of all black nodes is at least 8. By Lemma 3.5, we may assume that the degree of all white nodes in G_0 is at least three. We may also assume that G_0 is full, since deleting edges does not increase the degree of any node. We proceed by defining a black and white, plane graph G as the follows.

- If G_0 is 3-connected, let $G = G_0$.
- If G_0 is not 3-connected, then by Lemma 4.1, (1) and (5), G_0 is 2-connected and any node pair $\{b_1, b_2\}$ constituting a cut are black nodes. By Lemma 3.3, there exists a cut $\{b_1, b_2\}$ such that there is a component C of $G_0 - \{b_1, b_2\}$ such that $G_0[V(C) \cup \{b_1, b_2\}] + b_1b_2$ is 3-connected.

  Consider an embedding of $G_0[V(C) \cup \{b_1, b_2\}]$ defined by a restriction of the embedding of G_0. Clearly the edge b_1b_2 can be added without destroying planarity, so $G_0[V(C) \cup \{b_1, b_2\}] + b_1b_2$ is planar.

  Let $F_i, i = 1, 2$ be the two faces in $G_0[V(C) \cup \{b_1, b_2\}] + b_1b_2$ containing b_1b_2 in their boundaries. If one of the F_i is not a 3-face, then, for some i, $p = b(F_i) - b_1b_2$ is a path of length greater than two and is a part of the boundary of a face of G_0. But any face of G_0 either is a 3-face, in which case p is too long, or it is a 4-face with black and white nodes appearing alternately on the boundary, in which case b_1 and b_2 can not both be black. Therefore both the F_i are 3-faces.

  Let $b(F_i) = b_1a_ib_2b_1$. Suppose there is a degree three node u, other than a_1 and a_2, that is also adjacent to both b_1 and b_2. The edges b_1b_2, b_2u and ub_1 form a closed curve defining a regions which is not a face, by definition of a_1 and a_2. Hence a_1 and a_2 lie one internal and one external to this region. Since u is of degree three, there is no path between a_1 and a_2 that includes u, unless it also includes b_1 or b_2. Thus every path between a_1 and a_2 includes b_1 or b_2. Thus $\{b_1, b_2\}$ is a cut in $G_0[V(C) \cup \{b_1, b_2\}] + b_1b_2$, contradicting the fact that $G_0[V(C) \cup \{b_1, b_2\}] + b_1b_2$ is 3-connected. Hence there is no node of degree three, other than possibly a_1 and a_2, adjacent to both b_1 and b_2.

- If no more than one of a_1 and a_2 is a white node of degree three, let $G = G_0[V(C) \cup \{b_1, b_2\}] + b_1b_2$. By Lemma 4.1 (4), there is no more than one white node of degree three incident with exactly one 4-face, and there are no white nodes of degree three incident with less than one 4-face.
- If both a_1 and a_2 are white nodes of degree three let $G = G_0[V(C) \cup \{b_1, b_2\}] - b_1b_2$. Because a_1 and a_2 are adjacent to at least two 4-faces in G_0, by Lemma 4.1 (4), the same is true in G. We need to show that G is still 3-connected in the absence of b_1b_2.

  Suppose on the contrary that G is not 3-connected. Because $G - \{a_1, a_2\} + b_1b_2 = G_0[V(C) \cup \{b_1, b_2\}] + b_1b_2 - \{a_1, a_2\}$ is connected, $\{a_1, a_2\}$ must be a cut and $G - \{a_1, a_2\}$ has two components C_1, C_2 containing b_1, b_2 respectively. We may assume that a_1 is adjacent to only one node in C_1, namely b_1 since $d_G(a_1) = 3$.

  If $|C_1| \geq 2$, then $\{b_1, a_2\}$ would be a cut of $G + b_1b_2$, contradicting the fact that $G + b_1b_2 = G_0[V(C) \cup \{b_1, b_2\}] + b_1b_2$ is 3-connected. So it must be that $|C_1| = 1$. Then there must exist a black node b_3 such that $a_1b_3a_2b_1a_1$ forms the boundary of a 4-face of G_0 because a_1 and a_2 are not adjacent. Further, b_3 must be in C_2 and $b_3 \neq b_2$.

  If $|C_2| = 2$, then b_3 must be a black node of degree three in G_0, contradicting the our original assumption that all black node are of degree at least eight.

  If $|C_2| \geq 3$, it can be seen that $\{b_2, b_3\}$ is a cut in $G + b_1b_2$, contradicting the fact that this graph is 3-connected.

Summarising, we have that, for all three possible definitions, $G = (B \bigcup W, E)$ is a plane, reduced, black and white graph with the following properties:

(i) W is an independent set and $d_G(v) \geq 3$ for each $v \in W$.

(ii) G is 3-connected.

(iii) G satisfies (2) and (3) of Lemma 4.1.

(iv) There are at most two black nodes of degree less than eight.

(v) No white node of degree three is incident with less than one 4-face. At most one white node of degree three is incident with exactly one 4-face.

Let $b = |B|, w = |W|, v = |B \cup W|, e = |E|$. Let f and t_G denote the number of faces of G and the number of edge to triangulate G respectively. Let e_{bw} be the number of edges between black and white nodes in G, and let e_{bb} the number of edges between black nodes. Let $H = G[B]$, and let f_H, t_H and c_H denote the number of faces of H, the number of added edges necessary to triangulate H, and the number of components of H respectively, and let t_G be the number of added edges necessary to triangulate G. The following relationships are implied by these definitions:

$$\begin{aligned}
v &= b + w && (2)\\
e &= e_{bb} + e_{bw} && (3)\\
t_G &= 3v - 6 - e && (4)\\
t_H &= 3b - 6 - e_{bb} && (5)\\
2 &= v - e + f \quad \text{(by the Euler formula for } G\text{)} && (6)\\
c_H + 1 &= b - e_{bb} + f_H \quad \text{(by the Euler formula for } H\text{)} && (7)\\
f &= 2v - 4 - t_G \quad \text{(by (4) and (6))} && (8)
\end{aligned}$$

We know that G has $b - 2$ black nodes of degree at least 8 and, by property (iv), two black nodes of degree at least 2 and so:

$$8(b-2) + 4 \leq 2e_{bb} + e_{bw}.$$

By (3):

$$8b - 12 \leq e_{bb} + e \tag{9}$$

$$\begin{aligned}
c_H + 3 &= v + b - (e_{bb} + e) + f + f_H \quad \text{(by (6) and (7))} && (10)\\
&\leq v + b - 8b + 12 + f + f_H \quad \text{(by (9))} && (11)\\
&= 3v - 7b - t_G + f_H + 8 \quad \text{(by (8))} && (12)\\
&= 3w - 4b - t_G + f_H + 8 \quad \text{(by (2))} && (13)
\end{aligned}$$

Therefore:

$$3w - 4b - t_G + f_H - c_H \geq -5 \tag{14}$$

We continue by developing another argument that will contradict relation (14). By Lemma 4.1 (3), every 4-face of G contains exactly two white nodes, which are not adjacent since G is reduced. Let G' be the graph obtained from G by adding an edge between every pair of white nodes in the boundary of a 4-face. Then every face of G' is a 3-face. Let $L = G'[W]$, and denote by c_L and f_L the number of components and the number of faces of L respectively. H, defined as $G[B]$, is identical to $G'[B]$. H and L are plane subgraphs of G'.

All the nodes in any component of L lie (in G) within one face of H because, if not, there is an edge added between a pair of white nodes in G to form G' that crosses an edge of G. Let C_1 be a component of L and let T_1 be the set of 3-faces in G' whose boundary contains one or more black nodes and one or more nodes of C_1. Let C_2 be another component of L and let T_2 the set of 3-faces associated in the same way with C_2.

Suppose C_1 and C_2 lie in the same face of H. Then there is some 3-face F in both T_1 and T_2. Then either C_1 and C_2 share a node or there is a node from each component in F and there is an edge between them, since F is a 3-face. In either case, they are not separate components, Consequently, there can be no more than one component of L per face of H and we have:

$$c_L \leq f_H \tag{15}$$

The same argument can be inverted with respect to black and white, H and L, to yield a similar result:

$$c_H \leq f_L \tag{16}$$

Consider a particular triangulation of H produced by the following process and let H^T be the resulting graph.

Step 1: For each face 4-face of G, add an edge in the face joining the pair of black nodes in the boundary. Let G'' be the resulting plane graph.

Step 2: Remove all white nodes from G'' and triangulate each resulting faces that is not a triangle.

G is 3-connected, planar and satisfies property (iii). Consequently there is no edge in G between a pair of black nodes in a 4-face else that pair of black nodes would form a 2-node cut. Hence, G'' is a simple, triangulated, 3-connected, plane graph. Step 2 yields a triangulation of H.

By definition, t_G is equal to the number of edges added in Step 1, which is equal to the number of 4-faces of G, which is equal to $|E(L)|$, since the edges in L are exactly those added between each pair of white nodes in the 4-faces of G. Thus we have:

$$t_G = |E(L)| \tag{17}$$

By definition, t_H is equal to the sum the numbers of edges added in Steps 1 and 2 of the triangulation process. Let C be a component of L. Each edge of C corresponds to exactly one 4-face of G. In Step 1 of the triangulation, one edge is added for each 4-face of G. Therefore the number of edges in C is equal to the number of edges added in Step 1.

- If C is a not a tree, $|E(C)| \geq |V(C)|$. Therefore the number of edges added in Step 1 alone is at least $|V(C)|$.
- If C is a tree, then $|E(C)| = |V(C)| - 1$ and the number of edges added in Step 1 is at least $|V(C)|$ - 1. The tree C has at least two leaves. By property (v), for at least one leaf, say x, $d_G(x)$ can not be three, else it is in the boundaries of at least two 4-faces of G and it is not a leaf in C. Hence $d_G(x) \geq 4$, and so the face resulting from the removal of x in Step 2 contains at least four nodes in its boundary. Hence at least one edge is added in Step 2 of the triangulation process per component C. In total at least $|V(C)|$ edges are added in this case.

Thus the triangulation process adds at least $|V(C)|$ edges for each component C in L. Summing over all components of L, yields:

$$t_H \geq |V(L)| = w \tag{18}$$

Since $|V(L)| - |E(L)| + f_L = c_L + 1$ (Euler formula for L) and $w = |V(L)|$, we have

$$w - t_G + f_L = c_L + 1 \qquad \text{(by (17))} \tag{19}$$

$$\begin{aligned}
& 3w - 4b - t_G + f_H - c_H \\
= \; & 2w + (w - t_G + c_H) - 4b + f_H - 2c_H \\
= \; & 2w + (w - t_G + f_L) - 4b + f_H - 2c_H \qquad \text{(by (16))} \\
= \; & 2w + (c_L + 1) - 4b + f_H - 2c_H \qquad \text{(by (19))} \\
\leq \; & 2t_H - 4b + c_L + f_H - 2c_H + 1 \qquad \text{(by (18))} \\
= \; & 2(3b - 6 - e_{bb}) - 4b + c_L + f_H - 2c_H + 1 \qquad \text{(by (5))} \\
\leq \; & 2b - 2e_{bb} + f_H + f_H - 2c_H - 11 \qquad \text{(by (15))} \\
= \; & 2(b - e_{bb} + f_H - c_H) - 11 \\
= \; & -9 \qquad \text{(by (7))}
\end{aligned}$$

contradicting relation (14). □

5 The Bounded Genus Case

In this section we prove the following theorem for graphs of bounded genus.

Theorem 5.1 *If $G = (B \bigcup W, E)$ is a reduced black and white graph of genus at most $g \geq 1$, then there exists a node $b \in B$ such that $d_G(b) \leq 4g + 39$.*

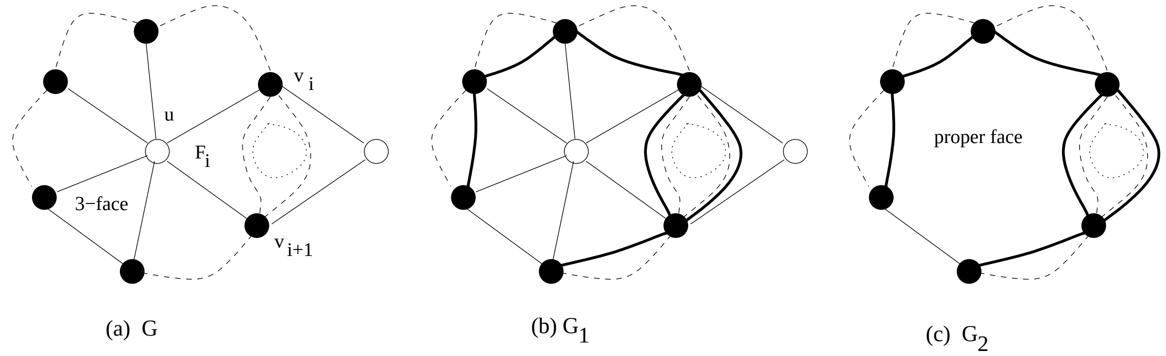

Figure 4: Embedding Operations

Proof

In contradiction to the theorem, suppose that $G = (B \bigcup W, E)$ is a reduced black and white graph with genus $\gamma(G) \leq g$ and that every node $u \in B$ is such that $d_G(u) \geq 4g + 40$. We may assume that G is a connected simple graph and that, by Lemma 3.5, for any $u \in W$, $d_G(u) \geq 3$. Consider G to be embedded in $\mathcal{S}_{\gamma(G)}$. Then the embedding is a 2-cell embedding by Lemma 3.1. By Lemmas 3.4 and 3.5, we have that $|B| \geq 3$. We consider two cases: either G is 3-connected or it is not.

G is 3-connected

First we show that $|B| \geq 6$. Suppose $3 \leq |B| \leq 5$. The maximum number of mutually non-inclusive subsets of five elements is $\left(\begin{array}{c} 5 \\ \lceil \frac{5}{2} \rceil \end{array} \right) = 10$, [Bol86, Chapter 3, Theorem 1]. But $|W| \geq 4g + 40 - 5 \geq 39$ and so there must exist two white nodes such that the neighbours of one of them constitute a subset of the neighbours of the other. But that contradicts R7 of the reduction procedure.

Let u be a white node of G, and let $v_0, v_1, \ldots, v_{t-1}$ be the sequence of nodes adjacent to u in clockwise order where $t = d_G(u)$. For i from 0 to $t-1$, let F_i be the face containing boundary edges $e_i = uv_i$ and $uv_{(i+1) \bmod t}$ in the current graph. If F_i is not a proper 3-face, then add an edge $e'_i = v_i v_{(i+1) \bmod t}$ so that e_i, e_{i+1}, e'_i form a proper 3-face.

Let us repeat this operation on all white nodes and denote the resulting embedded graph by G_1. Note that G_1 may contain multiple edges, but that no two multiple edges form the boundary of a 2-cell because G is a connected simple graph and the operation does not create a 2-face. Moreover, no two multiple edges are transformable in $\mathcal{S}_{\gamma(G)}$ since otherwise the two end nodes of two transformable edges would form a 2-node cut of G which contradicts our assumption that G is 3-connected. G_1 is uniquely determined, up to isomorphism, by the embedding of G.

Now let G_2 be the embedded graph obtained from G_1 by removing all white nodes and their incident edges. Then G_2 is a 2-connected 2-cell, embedded without 1-faces or 2-faces.

By construction, a white node of G_1 must be located in what becomes a proper face of G_2, and a proper face of G_2 contains at most one such white node in G_1. Since there are at most $d_{G_2}(x)$ faces incident with any node x in G_2, there are at most $d_{G_2}(x)$ white nodes adjacent to x in G_1 and hence

$$d_{G_1}(x) \leq 2d_{G_2}(x) \tag{20}$$

See Figure 4 for the construction of G_1 and G_2. Then $\gamma(G_2) \leq \gamma(G) \leq g$ and, by Lemma 3.2, we have

$$|E(G_2)| \leq 6(\gamma(G) - 1) + 3|B| \leq 6(g-1) + 3|B|.$$

Since $\delta(G_2)|B| \leq \sum_{u \in V(G_2)} d_{G_2}(u) = 2|E(G_2)|$, $\delta(G_2)|B| \leq 12(g-1) + 6|B|$ and hence $\delta(G_2) \leq 12(g-1)/|B| + 6 \leq 12(g-1)/6 + 6 = 2g+4$ and there exists $y \in B$ such that

$$d_{G_2}(y) \leq 2g + 4. \tag{21}$$

By inequality (20) we obtain

$$d_G(y) \leq d_{G_1}(y) \leq 2d_{G_2}(y) \leq 2(2g+4) < 4g + 40.$$

But this contradicts the assumption we used to begin the proof.

G is not 3-connected

By Lemma 3.3, G has a 2-node cut $\{x, y\}$ such that $G - \{x, y\}$ has a component C such that $G_0 = (V(C) \bigcup \{x, y\}, E(G[V(C) \bigcup \{x, y\}]) \cup \{xy\})$ is 3-connected. Since G contains a path connecting x and y which is internally disjoint with $V(C)$, the embedding of G induces an embedding of G_0 in $\mathcal{S}_{\gamma(G)}$ whether or not xy is an edge of G. Therefore $\gamma(G_0) \leq \gamma(G) \leq g$.

Let $B' = B \cap (V(C) \bigcup \{x, y\})$ and $W' = W \cap (V(C) \bigcup \{x, y\})$. We show that $|B'| \geq 8$. Since W is an independent set and $d_G(v) \geq 3, v \in W$, we must have $B' \setminus \{x, y\} \neq \emptyset$. Let $u \in B' \setminus \{x, y\}$, then $d_{G_0}(u) = d_G(u) \geq 4g + 40 \geq 44$. Then we have $|B'| + |W'| \geq 45$. If $|B'| \leq 7$, then $|W'| \geq 38$ and $|W' \setminus \{x, y\}| \geq 36$. The neighbours of nodes in $W' \setminus \{x, y\}$ are in B', and so there must exist two white nodes such that the neighbours of one of them constitute a subset of the neighbours of the other. This is because the maximum number of mutually non-inclusive subsets of seven elements is $\binom{7}{\lceil \frac{7}{2} \rceil} = 35 < 36$. But this would contradict R7 of the reduction procedure.

Now let $B_0 = B' \cup \{x, y\}$ be the black node set for G_0 and consider G_0 to be embedded in $\mathcal{S}_{\gamma(G_0)}$. Let G_1 and G_2 be derived from G_0 by exactly the same process as described in the previous case. The inequality (20) still holds.

By Lemma 3.2, we have

$$|E(G_2)| \leq 6(\gamma(G_2) - 1) + 3|B_0| \leq 6(g-1) + 3|B_0|.$$

Since $2|E(G_2)| = \sum_{u \in V(G_2)} d_{G_2}(u) \geq \delta(G_2)|B_0|$ and $|B_0| \geq |B'| \geq 8$, there is at least one black node $z \in B_0 \setminus \{x, y\}$ such that

$$d_{G_2}(z) \leq \frac{12(g-1) + 6|B_0|}{|B_0| - 2} = \frac{12g}{|B_0| - 2} + 6 \leq \frac{12g}{6} + 6 = 2g + 6 \tag{22}$$

Then, by inequalities (22) and (20), we have

$$d_G(z) = d_{G_0}(z) \leq d_{G_1}(z) \leq 2d_{G_2}(z) \leq 2(2g+6) < 4g + 39.$$

But this contradicts the assumption we used to begin the proof. □

Combining the degree bound $4g + 39$ of Theorem 5.1 with the time complexity of $O((d+1)^k t(n))$, where $t(n) = O(n^2)$, for the basic algorithm we obtain our claimed time complexity of $O((4g+40)^k n^2)$.

6 Conclusions

In Section 2 we showed a time complexity of $O(n^2)$ for the reduction procedure. In Section 5 we showed that in any reduced black and white graphs of bounded genus g there exists at least one node of degree at most $4g+39$. Combining these results yields a time complexity of $O((4g+40)^k n^2)$ for the basic algorithm, after the reduction procedure has been incorporated.

It would be interesting to know if other results known for planar graphs, for example, the existence of a linear kernel and the $O(2^{O(\sqrt{k})} n^c)$ algorithm via tree decomposition, can be extended to graphs of bounded genus.

Acknowledgment

We are indebted to the referee for valuable suggestions that have improved the result.

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