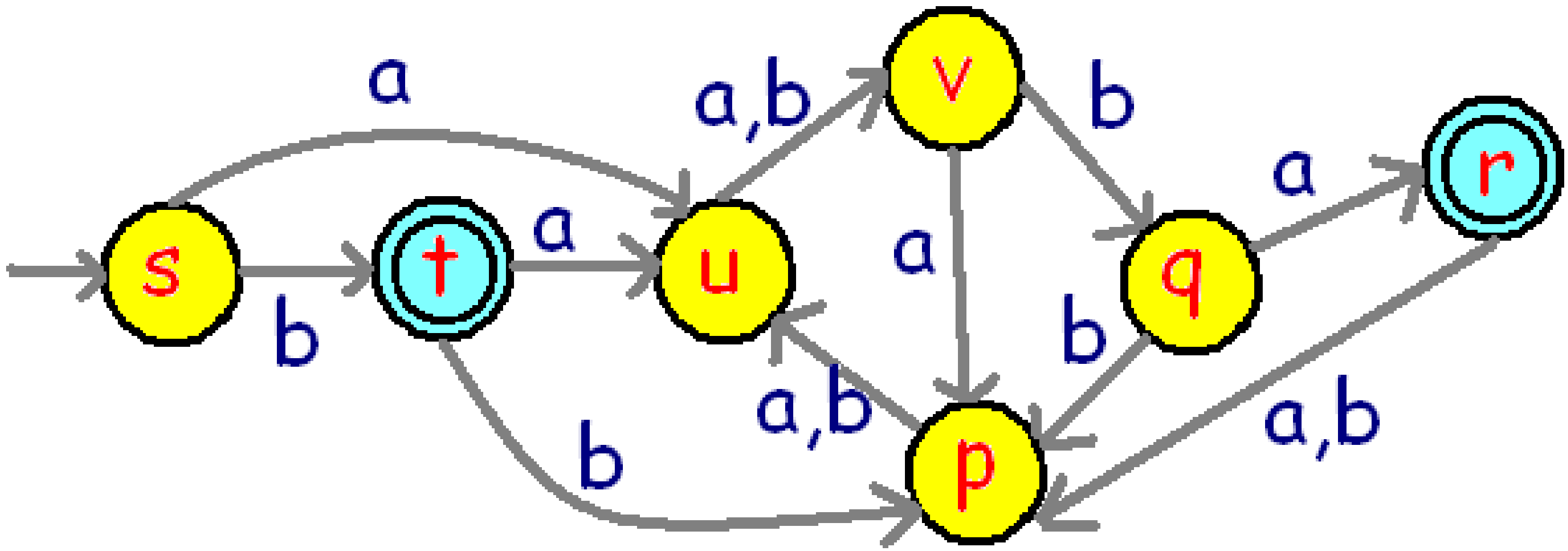
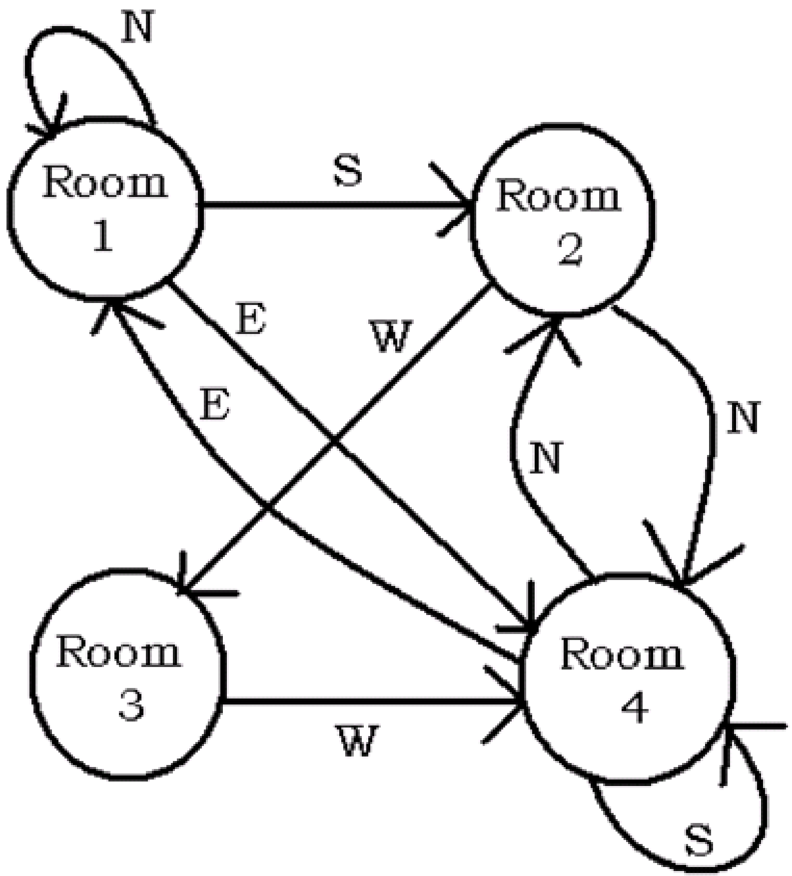
Find as many examples as you can of w, x, y, z so that w is accepted by this DFA, $w = x\,y\,z$, $y \neq \varepsilon$, $|\,x\,y\,| \leq 7$, and $x\,y^n\,z$ is in L for all $n \geq 0$.



You are in a maze of twisty little passages, all alike.

If the maze has n rooms and each one has trails exiting to the N, S, W, E. How many trails must be traversed before some room is visited more than once?



The Pigeonhole Principle

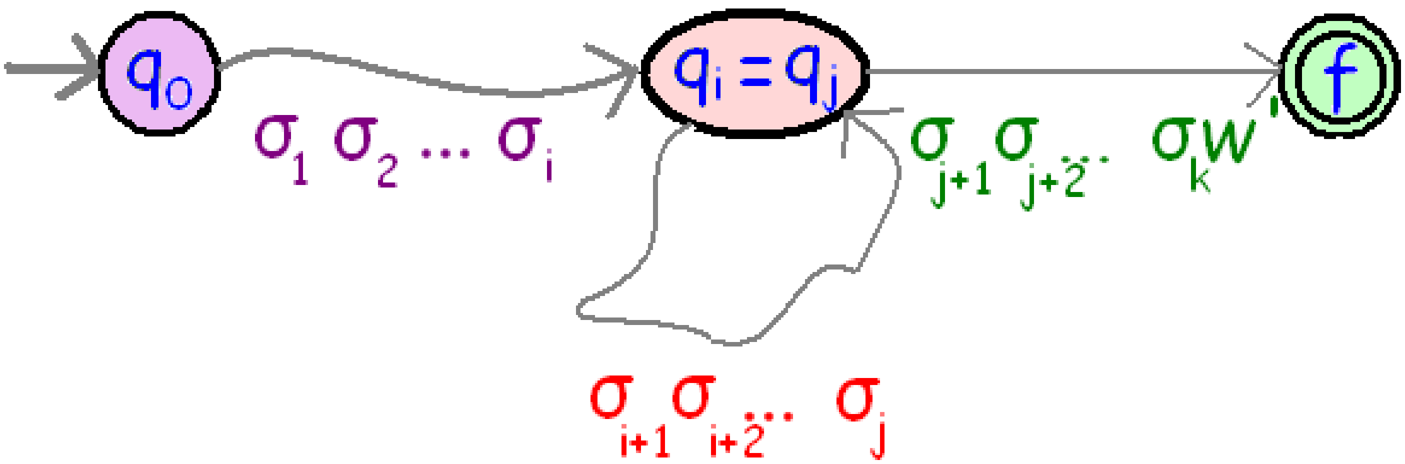
Given two natural numbers n and m with $n > m$, if n items are put into m pigeonholes, then at least one pigeonhole must contain more than one item.

Picture from: Wikipedia, the free encyclopedia

Proof of the Pumping Lemma

The pumping lemma is a statement about regular languages used to show using a proof by contradiction that languages are not regular.

The goal today is to prove it and show further illustrations of how it can be used.



The Pumping Lemma for Regular Languages:

If L is a language accepted by a DFA with k states, and $w \in L$, $|w| \geq k$, then there exists x, y, z such that

1. $w = x\,y\,z$,
2. $y \neq \varepsilon$,
3. $|\,x\,y\,| \leq k$, and
4. $x\,y^n\,z$ is in L for all $n \geq 0$.

Proof. Consider $w \in L$, $|w| \geq k$.

Isolate the first k symbols in w:

$w = \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_{k-1} \sigma_k w'$.

Consider the computation of M on w:

$(q_0, \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_{k-1} \sigma_k w') \vdash$

$(q_1, \sigma_2 \sigma_3 \ldots \sigma_{k-1} \sigma_k w') \vdash$

$(q_2, \sigma_3 \ldots \sigma_{k-1} \sigma_k w') \vdash^*$

$(q_{k-1}, \sigma_k w') \vdash^*$

$(q_k, w') \vdash^* (f, e)$ for some final state f.

After reading $\sigma_1\ \sigma_2\ \sigma_3\ \ldots\ \sigma_{k-1}\ \sigma_k$

k+1 states have been visited: $q_0, q_1, q_2, \ldots q_k$.

The computation:

$(q_0, \sigma_1\ \sigma_2\ \sigma_3\ \ldots\ \sigma_{k-1}\ \sigma_k\ w') \vdash$

$(q_1, \sigma_2\ \sigma_3\ \ldots\ \sigma_{k-1}\ \sigma_k\ w') \vdash$

$(q_2, \sigma_3\ \ldots\ \sigma_{k-1}\ \sigma_k\ w') \vdash^*$

$(q_{k-1}, \sigma_k\ w') \vdash^*$

$(q_k, w') \vdash^* (f, e)$ for some final state f.

By the pigeonhole principle, some state has been repeated.

By the pigeonhole principle, some state has been repeated. Suppose q_i is the first time we see the repeated state and q_j is the second time.

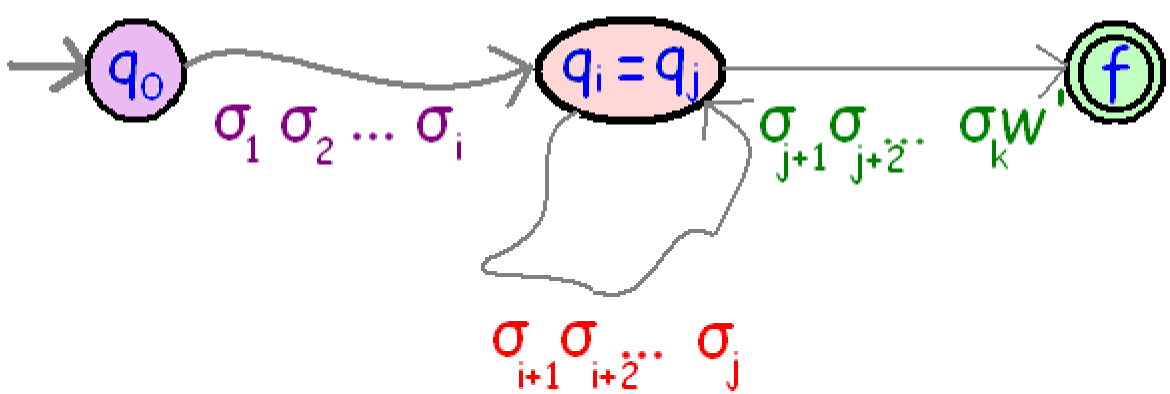
Rewriting the computation:

$(q_0, \sigma_1\, \sigma_2\, \sigma_3 \ldots \sigma_i\, \sigma_{i+1} \ldots \sigma_j\, \sigma_{j+1} \ldots \sigma_{k-1}\, \sigma_k\, w') \vdash^*$

$(q_i, \sigma_{i+1} \ldots \sigma_j\, \sigma_{j+1} \ldots \sigma_{k-1}\, \sigma_k\, w') \vdash^*$

$(q_j, \sigma_{j+1} \ldots \sigma_{k-1}\, \sigma_k\, w') \vdash^* (f, e)$

for some final state f.



Digression from Proof:

$(q_0, \sigma_1\, \sigma_2\, \sigma_3\, \ldots\, \sigma_i\, \sigma_{i+1}\, \ldots\, \sigma_j\, \sigma_{j+1}\, \ldots\, \sigma_{k-1}\, \sigma_k\, w')\ \vdash^*$

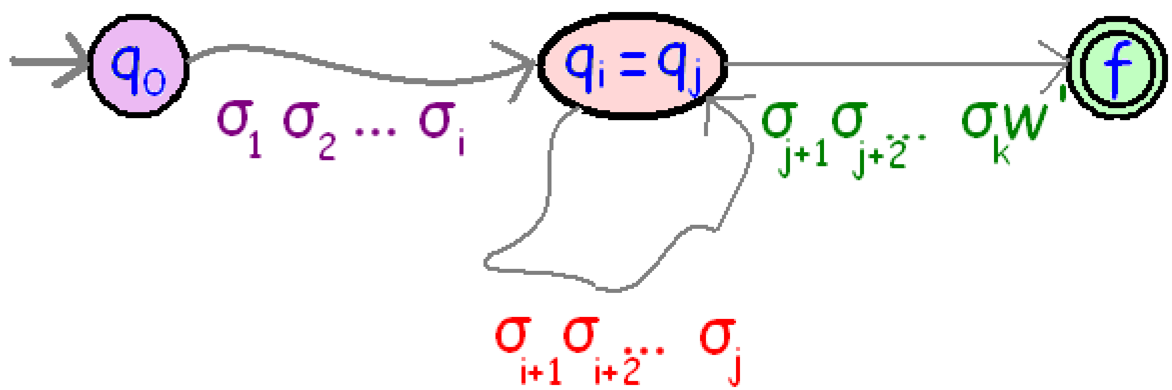
$(q_i, \sigma_{i+1}\, \ldots\, \sigma_j\, \sigma_{j+1}\, \ldots\, \sigma_{k-1}\, \sigma_k\, w')\ \vdash^*$

$(q_j, \sigma_{j+1}\, \ldots\, \sigma_{k-1}\, \sigma_k\, w')\ \vdash^* (f, e)$

We can pump 0 times because:

$(q_0, \sigma_1\, \sigma_2\, \sigma_3\, \ldots\, \sigma_i\, \sigma_{j+1}\, \ldots\, \sigma_{k-1}\, \sigma_k\, w')\ \vdash^*$

$(q_i = q_j, \sigma_{j+1}\, \ldots\, \sigma_{k-1}\, \sigma_k\, w')\ \vdash^* (f, e)$



Digression from Proof:

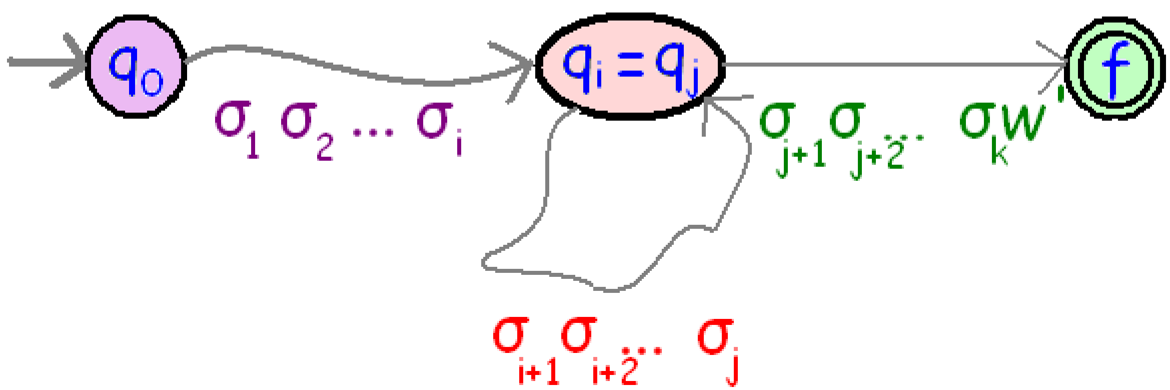
We can pump 2 times because:

$(q_0, \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_i \ (\sigma_{i+1} \ldots \sigma_j)^2 \ \sigma_{j+1} \ldots \sigma_k \ w' \ ) \ \vdash^*$

$(q_i, (\sigma_{i+1} \ldots \sigma_j)^2 \ \sigma_{j+1} \ldots \ \sigma_k \ w' \ ) \ \vdash^*$

$(q_j = q_i, (\sigma_{i+1} \ldots \sigma_j) \ \sigma_{j+1} \ldots \ \sigma_k \ w' \ ) \ \vdash^*$

$(q_j, \sigma_{j+1} \ldots \ \sigma_k \ w' \ ) \ \vdash^* (f, e)$



Let $x = \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_i$, $y = \sigma_{i+1} \ldots \sigma_j$, and

$z = \sigma_{j+1} \ldots \sigma_k w'$. The string $x\, y^n\, z$ is in L for all values of $n \geq 0$ since

$(q_0, \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_i\ (\sigma_{i+1} \ldots \sigma_j)^n \sigma_{j+1} \ldots \sigma_k w')\ \vdash^*$

$(q_j = q_i, (\sigma_{i+1} \ldots \sigma_j)^n \sigma_{j+1} \ldots \sigma_k w')\ \vdash^*$

$(q_j = q_i, (\sigma_{i+1} \ldots \sigma_j)^{n-1} \sigma_{j+1} \ldots \sigma_k w')\ \vdash^*$

...

$(q_j = q_i, (\sigma_{i+1} \ldots \sigma_j) \sigma_{j+1} \ldots \sigma_k w')\ \vdash^*$

$(q_j, \sigma_{j+1} \ldots \sigma_k w')\ \vdash^* (f, e)$.

End of Proof.

The Pumping Lemma for Regular Languages:

If L is a language accepted by a DFA with k states, and $w \in L$, $|w| \geq k$, then there exists x, y, z such that

1. $w = x\,y\,z$,
2. $y \neq \varepsilon$,
3. $|\,x\,y\,| \leq k$, and
4. $x\,y^n z$ is in L for all $n \geq 0$.

$L = \{ a^n b^n : n \geq 0 \}$ is not a regular language.

Proof (by contradiction)

Assume L is regular.

Then L is accepted by a DFA M with k states for some integer k.

Since L is accepted by a DFA M with k states, the pumping lemma holds.

Let $w = a^r b^r$ where $r = \lceil k/2 \rceil$.

Consider all possibilities for y:

Case 1: $a^i \ (a^j)\ a^{r-i-j}\ b^r \qquad j \geq 1.$

Pump zero times: $a^{r-j}\ b^r \notin L$ since $r-j < r$.

Case 2: $a^r\ b^i \ (b^j)\ b^{r-i-j} \qquad j \geq 1.$

Pump zero times: $a^r\ b^{r-j} \notin L$ since $r-j < r$.

Case 3: $a^{r-i}\ (\ a^i \ \ b^j\ )\ \ b^{r-j} \qquad i, j \geq 1.$

Pump 2 times: $a^{r-i} a^i \ \ b^j \ \ a^i \ \ b^j \ \ b^{r-j} \notin L$
because it is not of the form $a^* b^*$

Therefore, L is not regular.

Using closure properties:

$L = \{w \in \{a, b\}^* : w$ has the same number of a's as b's$\}$ is not regular.

Proof (by contradiction)

Assume L is regular. The language $a^* b^*$ is regular since it has a regular expression. Because regular languages are closed under intersection, $L \cap a^* b^*$ is regular. But

$L \cap a^* b^* = \{ a^n b^n : n \geq 0\}$ which is not regular.

Therefore, L is not regular.

Problem of the Day (next class):

Factor $(ab)^k$ as xyz in all ways such that $y \neq \varepsilon$.