CSC 225 Midterm Exam

Thurs. Oct. 22, 1992

1. [10] What does it mean for an algorithm to be worst case optimal? Define **all** terms used in your definition [4 marks each].

2. [10] Prove by induction that

$$\sum_{i=1}^{n} i\, 2^i \;=\; (n-1)\, 2^{n+1} + 2, \quad \textit{for all } n \geq 1.$$

3. [30] Let $n = \sum_{i=1}^{k} i$, $k \geq 1$.

   (a) [5] State a closed formula for n in terms of k (you do not need to justify your answer).

   (b) [5] Prove that $k/2$ is a lower bound for $\sqrt{n}$.

   (c) [5] Prove that $k+1$ is an upper bound for $\sqrt{n}$.

   (d) [10] State a definition of the set of functions $\Theta(f(n))$. Do not use limits in your definition. (You can define it in terms of O and/or Ω if you also define these without using limits).

   (e) [5] Use your definition from (d) and parts (b) and (c) to prove that $k \in \Theta(\sqrt{n})$.

Questions 4-6 refer to the following problem.

You wish to determine the height expressed in floors of a building from which a Little Tykes tricycle will break if tossed out of a window on that floor. The building has n floors, and k tricycles are available. You are allowed only one operation to determine the breaking height- you may toss a tricycle out the window. If it does not break it can be reused in another experiment.

One possible outcome is that tricycles will not break even when dropped from the top floor. Otherwise there is some floor f so that any tricycle dropped from a floor below f does not break and a tricycle dropped from floor f or higher does break. The fact that a tricycle has been used in a previous experiment (and it did not break) does not affect the breaking height of the tricycle.

4.(a) [4] Given that the probability that the tricycle does not break when dropped from any of the floors is 5/7 and that all other possible outcomes of the experiment are equally likely, what is the probability that the tricycle breaks when dropped from floor i?

(b) [6] If there is only one tricycle available, then a linear search starting at the first floor is the only strategy that works. Compute the average number of experiments needed to determine the breaking height of a tricycle using this linear search, with the assumptions from (a).

4(c) [10] If the number of tricycles is unlimited, then the strategy that minimizes the number of experiments is a binary search. How many tricycles break in the worst case? Describe this worst case situation and give an exact formula for the number of tricycles broken in terms of n the number of floors of the building. Assume that the midpoint is chosen for the sub-problem for floors *lower* to *upper* by the formula $mid = \left\lfloor \frac{lower + upper}{2} \right\rfloor$.

5. Given k tricycles, we propose the following strategy for this problem. We do binary search as in Question 4(c) until only one tricycle remains. Then a linear search is used as described in Question 4(b).

(a) [10] Draw the decision tree that represents the experiments done by this algorithm when $n = 10$, and $k = 2$. The left branch should represent the case where a tricycle breaks and the right branch when it does not break. Label the nodes with the number of the floor that a tricycle is dropped from.

(b) [3] What is the worst case complexity of this algorithm in terms of the number of experiments completed when $n = 10$ and $k = 2$?

(c) [7] What is the worst case complexity of this algorithm (number of experiments) for arbitrary n when $k = 2$. Justify your answer by describing the worst case scenario.

6. [10] A student who has not yet taken this course argues that the strategy suggested for Question 5 must be optimal (minimum number of experiments) in the worst case since from 4(b) linear search is required when there is only one tricycle and from 4(c) binary search is the optimal strategy when there is no shortage of tricycles. Prove that this argument is incorrect by providing a decision tree for $n = 10$ and $k = 2$ which corresponds to an algorithm with better worst case complexity.

7. [10- Bonus] Provide an algorithm for $k = 2$ which is asymptotically better than the approach suggested in Question 4. Give a general description of the decision trees for your improved algorithm. Analyse your algorithm to find its worst case complexity.