Which n-Venn diagrams can be drawn with convex k-gons?

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Abstract

We establish a new lower bound for the number of sides required for the component curves of simple Venn diagrams made from polygons. Specifically, for any n-Venn diagram of convex k-gons, we prove that $k \geq (2^n - 2 - n)/(n(n-2))$. In the process we prove that Venn diagrams of seven curves, simple or not, cannot be formed from triangles. We then give an example achieving the new lower bound of a (simple, symmetric) Venn diagram of seven convex quadrilaterals. Previously Grünbaum had constructed a symmetric 7-Venn diagram of non-convex 5-gons ["Venn Diagrams II", *Geombinatorics* 2:25-31, 1992].

1 Introduction and Background

Venn diagrams and their close relatives, the Euler diagrams, form an important class of combinatorial objects which are used in set theory, logic, and many applied areas. Convex polygons are fundamental geometric objects that have been investigated since antiquity. This paper addresses the question of which convex polygons can be used to create Venn diagrams of certain numbers of curves. This question has been studied over several decades, for example [1, 2, 7, 8, 9]. See the on-line survey [12] for more information on geometric aspects of Venn diagrams.

Let $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ be a family of n simple closed curves in the plane that are finitely intersecting. We say that $\mathcal{C}$ is a *Venn diagram* (or *n-Venn diagram*) if all of the 2^n open regions $X_1 \cap X_2 \cap \cdots \cap X_n$ are non-empty and connected, where each set X_i is either the bounded interior or the unbounded exterior of the curve C_i. If the connectedness condition is dropped the diagram is called an *independent family*. We can also think of the diagram as a plane edge-coloured graph whose vertices correspond to intersections of curves, and whose edges correspond to the segments of curves between intersections. Edges are coloured according to the curve to which they belong. A Venn diagram or

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independent family is *simple* if at most two curves intersect at a common point, *i.e.* every vertex has degree exactly four. Two diagrams are considered *isomorphic* if one can be transformed into the other, or its mirror image, by a continuous deformation of the plane.

A polygon is a simple closed curve composed of finitely many line segments. We refer to those line segments as *sides* and the intersections of sides as *corners*. Let the term *k-gon* designate a *convex* polygon with exactly k sides. Observe that two k-gons can (finitely) intersect with each other in at most $2k$ points.

In this paper, we consider Venn diagrams and independent families composed of k-gons, for some k. Note that an edge of a k-gon, using the graph interpretation of a Venn diagram, may contain zero or more corners of the k-gon containing that edge. Note also that the corners of the component k-gons are not vertices in the graph interpretation of any diagram unless they intersect another curve at that corner. It will prove convenient, when considering simple diagrams, to assume that different curves do not intersect at corners.

Lemma 1.1. *Every simple Venn diagram whose curves are polygons is isomorphic to one in which curves do not intersect at corners.*

Proof. Given a simple Venn diagram, the set of intersection points and corners is a finite set, call it $\mathcal{I}$. Now suppose that curve C has a corner x that intersects one other curve and let $s_1 = (y, x)$ and $s_2 = (x, z)$ be two of the sides of C. Define a new point x' that is "close" to x and on s_1, x' and is such that the triangle whose vertices are $\{x, x', z\}$ does not contain any of the points of $\mathcal{I}$ other than x and z.

The diagram obtained by replacing s_1 and s_2 by $s_1' = (y, x')$ and $s_2' = (x', z)$ is still a Venn diagram since no regions are created or destroyed, and it is isomorphic to the original diagram. Furthermore, it has one fewer corner intersecting other curves so the result follows by induction. □

The proof of Lemma 1.1 does not apply in the non-simple case since extra regions could be created by replacing s_2 by s_2' if C intersected more than one other curve at some point on s_2. By Lemma 1.1, it is clear that if a simple n-Venn diagram can be drawn with k-gons, it can also be drawn with j-gons for any $j > k$, by adding small sides at corner points.

Henderson [10] gave an example of a symmetric Venn diagram of five quadrilaterals. Grünbaum first considered the problem of what polygons can be used to create Venn diagrams in [7], in which he gave a Venn diagram of six quadrilaterals (4-gons), a diagram of five triangles, and an independent family of five squares. He also provides a construction showing that n-Venn diagrams can be constructed from convex polygons, for any n. In [9] he conjectured that there is no symmetric Venn diagram of five squares.

We restate two lemmas first observed by Grünbaum [7], some of the consequences of which inspired this work. A *FISC* is a family of *F*initely *I*ntersecting *S*imple closed

*C*urves in the plane, with the property that the intersection of the interiors of all the curves is not empty [3]. Clearly, every Venn diagram is a FISC.

Lemma 1.2. *In a FISC of n convex k-gons there are at most $\binom{n}{2}2k$ vertices.*

Proof. A pair of convex k-gons can intersect with each other at most $2k$ times; there are $\binom{n}{2}$ pairs. □

Lemma 1.3. *In a simple n-Venn diagram of k-gons,*

$$k \geq \left\lceil (2^{n-1}-1)/\binom{n}{2} \right\rceil . \tag{1}$$

Proof. Euler's formula for plane graphs, combined with the fact that in a simple diagram all vertices have degree four, implies that the number of vertices in a simple Venn diagram is $2^n - 2$. Combining this with Lemma 1.2, which gives an upper bound on the number of vertices, the inequality follows. □

Lemma 1.3 gives us a bound, for each n, on the minimum k required to form a simple n-Venn diagram of k-gons. Diagrams are well-known that achieve the bounds for $n \leq 5$; see [12] for examples. For $n = 6$, the Lemma implies $k \geq 3$, and Carroll [4] achieved the lower bound by giving examples of 6-Venn diagrams formed of triangles; his diagrams are all simple. Figure 1 shows one of Carroll's Venn diagrams of six triangles.

For $n = 7$, Lemma 1.3 implies that $k \geq 3$; however until now the diagram with the smallest known k was a 7-Venn diagram of non-convex 5-gons by Grünbaum in [9].

The contributions of this paper are, first, to prove a tighter lower bound than Lemma 1.3 for the minimum k required to draw a simple Venn diagram of k-gons; second, to show that no 7-Venn diagram of triangles (simple or not) can exist, and third, to achieve the new lower bound for $n = 7$ by exhibiting a Venn diagram of seven quadrilaterals.

In [7], Theorem 3 contains bounds on $k^*(n)$, which is defined as the minimal k such that there exists a Venn diagram of n k-gons. Carroll's results prove that $k^*(6) = 3$, and our results prove that $k^*(7) = 4$, and provide a lower bound on $k^*(n)$ for $n > 7$ when considering simple diagrams.

2 Venn Diagrams of k-Gons

We now prove a tighter lower bound than that given by Lemma 1.3 for simple Venn diagrams.

Observation 2.1. *In a Venn diagram composed of convex curves, each curve has exactly one edge on the outer face.*

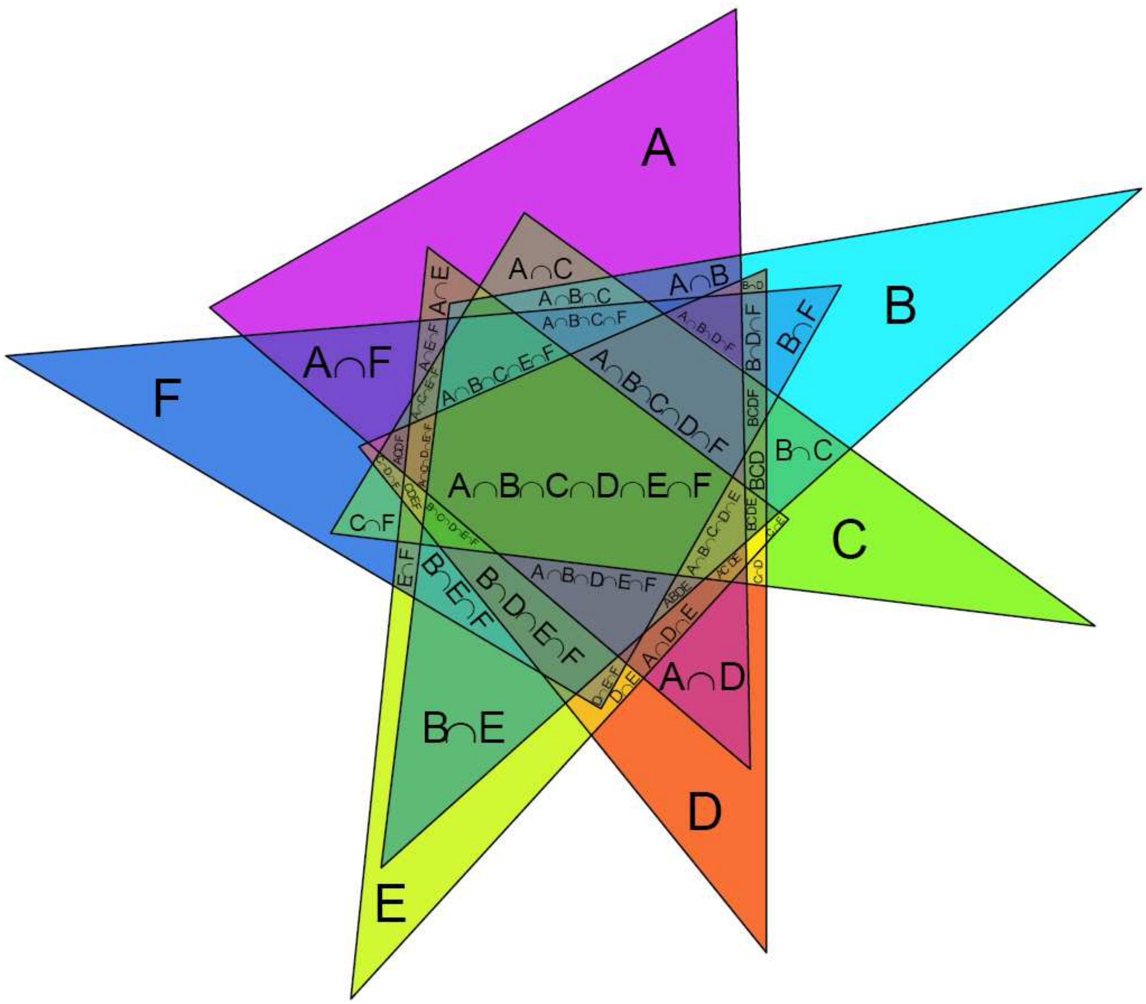

Figure 1: A Venn diagram of six triangles.

Proof. An r-region is a region contained inside exactly r curves. It is proven in [3] that in a Venn diagram composed of convex curves, every r-region with $r > 0$ is adjacent to an $(r-1)$-region. In particular, every 1-region is adjacent to the outer face, and thus every curve has an edge on the outer face. Furthermore, Lemma 4.6 from [5] states that no two edges on any face in Venn diagram can belong to the same curve. □

We now introduce some notation before proving the main theorems of this section. In a Venn diagram of k-gons, consider any two k-gons C_i and C_j, $1 \le i < j \le n$. The corners of C_i may be labelled according to whether they are external (E) to C_j or internal (I) to C_j We only consider simple diagrams here and thus by Lemma 1.1 we can assume that curves do not intersect at corners.

In a clockwise walk around C_i we obtain a circular sequence of k occurrences of E or I. Let EI_{ij} denote the number of occurrences of an E label followed by an I label; the notations II_{ij} and IE_{ij} are defined in an analogous manner. In the situation that an E follows an E we distinguish two cases: either C_i is intersected twice on the side between the two E corners, or it is not intersected. The notation EE_{ij} is for the case where no intersection with C_j occurs and EE'_{ij} for the case where two intersections occur. By convexity, C_i can only be intersected at most twice in a side by C_j. Since these cases cover all types of corners,

$$EI_{ij} + IE_{ij} + II_{ij} + EE_{ij} + EE'_{ij} = k. \tag{2}$$

Also note that $EI_{ij} = IE_{ij}$ since there must be an even number of crossings between the curves.

Theorem 2.2. *In a simple Venn diagram of k-gons,*

$$V \le 2k\binom{n}{2} - n(k-1)\ .$$

Proof. Given the notation above, consider the entire collection of curves. Label corners on the outer face ϵ and the others ι. Define E_i to be the number of corners of C_i labelled ϵ and I_i to be the number labelled ι. Clearly $I_i + E_i = k$.

In a Venn diagram each of the n k-gons has one outer edge, by Observation 2.1, and so all corners of C_i labelled ϵ must appear contiguously; thus

$$\sum_{i \ne j} EE_{ij} \ge \sum_i (E_i - 1) \tag{3}$$

since the left-hand term will also count corners external to some curve but internal to others.

Since any corner labelled ι is internal to some curve,

$$\sum_{i \neq j} (II_{ij} + IE_{ij}) \geq \sum_i I_i \tag{4}$$

since the left-hand term will double count any corner on C_i internal to more than one curve.

Since each EI and IE accounts for one intersection and EE' for two intersections,

$$\begin{aligned}
2V &= \sum_{i \neq j} (EI_{ij} + IE_{ij} + 2EE'_{ij}) \\
&= \sum_{i \neq j} (2k - 2II_{ij} - 2EE_{ij} - EI_{ij} - IE_{ij}) && \text{by (2).} \\
&= 4k\binom{n}{2} - \sum_{i \neq j} (2II_{ij} + 2EE_{ij} + 2EI_{ij}) \\
&\leq 4k\binom{n}{2} - 2\sum_i (E_i - 1) - 2\sum_i I_i && \text{by (3) and (4).} \\
&= 4k\binom{n}{2} - 2\sum_i (E_i - 1 + I_i) \\
&= 4k\binom{n}{2} - 2\sum_i (k - 1) \\
&\leq 4k\binom{n}{2} - 2n(k - 1).
\end{aligned}$$

Dividing by 2 gives $V \leq 2k\binom{n}{2} - n(k-1)$, as desired. □

Theorem 2.3. *In any simple n-Venn diagram of k-gons,*

$$k \geq \left\lceil \frac{2^n - 2 - n}{n(n-2)} \right\rceil .$$

Proof. For simple Venn diagrams, we have that the number of vertices is $2^n - 2$. Combined with Theorem 2.2, we have

$$\begin{aligned}
2^n - 2 &\leq 2k\binom{n}{2} - n(k-1) \\
&= n(n-1)k - nk + n .
\end{aligned}$$

$$\text{Thus} \quad 2^n - 2 - n \leq k(n(n-1) - n)$$

$$\text{or} \quad \left\lceil \frac{2^n - 2 - n}{n(n-2)} \right\rceil \leq k,$$

as desired. □

Theorem 2.3 gives a lower bound on the minimum k required to construct a simple n-Venn diagram of k-gons. Table 1 shows the bound for small values of n.

For an upper bound on k, note that there are many general constructions for Venn diagrams that produce diagrams of k-gons where the value k is a function of n (for examples, see [11] or [7]). In Grünbaum's convex construction in [7], the nth curve is a convex 2^{n-2}-gon; this gives the upper bounds in Table 1 for $n > 7$. Including this paper's contributions, diagrams are known for $n \leq 7$, thus solving these cases.

n	3	4	5	6	7	8	9	10	11	12	13	14
$k \geq$	1	2	2	3	4	6	8	13	21	35	58	98
$k \leq$	1	2	2	3	4	64	128	256	512	1024	2048	4096

Table 1: Minimum k required to construct a simple n-Venn diagram of k-gons.

3 7-Venn Diagrams of Triangles

In this section we prove that there is no 7-Venn diagram, simple or not, composed of triangles. The bound in Theorem 2.3, for $n = 7$, gives $k \geq 4$, which proves the simple case.

In a non-simple diagram, there must exist at least one vertex where at least three curves intersect. This vertex can be reduced in degree by the operation of *splitting* the vertex; we now define this operation in terms of moving one of the curves away from the vertex. Consider a polygon P with corners $(p_1, p_2, \ldots, p_k)$ in clockwise order, such that a vertex v of degree at least six lies on the side (p_i, p_{i+1}).

As a preliminary step we require that no high-degree vertices lie on corners of P; see Figure 2 for an example in which the vertex v is on the corner p_i. To eliminate this situation, define the ray L emanating from p_{i-1} containing (p_{i-1}, p_i) and define p'_i on L, lying "close" (in the sense of the proof of Lemma 1.1) to p_i but not on (p_{i-1}, p_i). Replacing p_i by p'_i in P ensures that v is now on the side (p_{i-1}, p'_i) but not on a corner of P. Note that no regions are destroyed as all of the faces of which v was on the border are still present. Some faces may be created as a result of this operation, and any high-degree vertices on the side (p_i, p_{i+1}) may be split as a result (as in Figure 3).

Thus by induction we can assume that no vertices in the diagram lie on corners; at this point the diagram may no longer be a Venn diagram but it is an independent family. Now consider a high-degree vertex v on the side (p_i, p_{i+1}); see Figure 3. To split v, replace p_i in P with p'_i, where p'_i is "close" to p_i such that p'_i is on the side (p_{i-1}, p_i) and close enough to p_i that (p'_i, p_{i+1}) intersects no other vertices of the diagram. This operation alters P by replacing the sides $(p_{i-1}, p_i), (p_i, p_{i+1})$ with $(p_{i-1}, p'_i), (p'_i, p_{i+1})$; the side (p'_i, p_{i+1}) does not intersect v as it is distinct from (p_i, p_{i+1}) at every point except

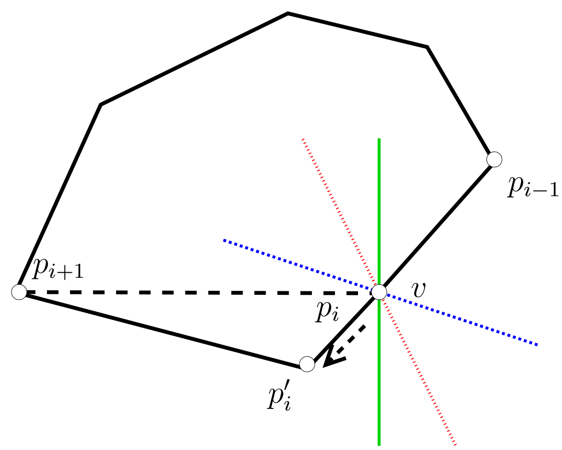

Figure 2: Moving corners away from high-degree vertex v as a preliminary step.

p_{i+1}. Note that splitting vertex v may happen to split additional vertices along the side (p_i, p_{i+1}).

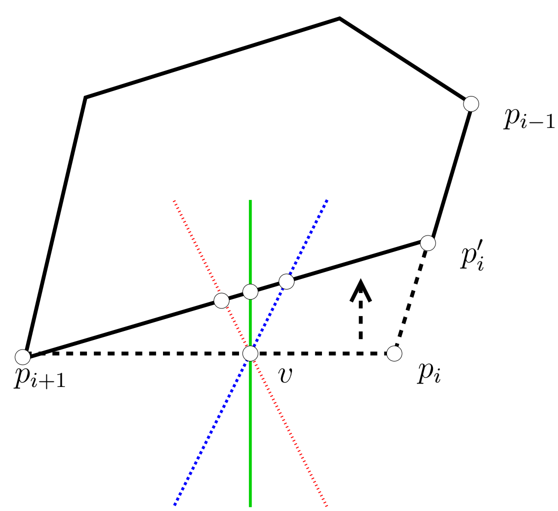

Figure 3: Splitting a vertex v composed of intersecting straight-line segments.

Splitting a vertex can never remove a face: all of the faces where v was on the border are still present after the translation, though some have become smaller; furthermore, the operation must add at least one face to the resulting diagram. Thus, after splitting any large degree vertex, the resulting diagram will no longer be a Venn diagram as some face must be duplicated, but it will still be an independent family.

By induction, all large-degree vertices can be moved from corners and then split repeatedly to form several vertices of degree four. We use this operation to prove the following Lemma.

Lemma 3.1. *There is no non-simple Venn diagram of seven triangles.*

Proof. Assume such a diagram exists; call it D_0. Since D_0 is non-simple, some vertices have degree greater than four. Let D_1 be the simple independent family formed by splitting all of the high-degree vertices in D_0. This can be performed while still retaining the fact that D_1 is composed of triangles, by incrementally translating the edges of the component triangles as described above. Let F_i, E_i, and V_i be the number of faces, edges, and vertices in D_i, for $i \in \{0, 1\}$.

Since D_0 is a Venn diagram, $F_0 = 128$, and $F_1 > F_0$ since some new faces must have been created by splitting vertices to form D_1. Since D_1 has all degree-four vertices, summing the vertex degrees gives us $E_1 = 2V_1$. Using Euler's formula, $V_1 + F_1 - E_1 = 2$, and substituting for E_1 gives $V_1 = F_1 - 2 > F_0 - 2 = 126$, and so $V_1 > 126$.

However, D_1 is composed of triangles, two of which can only intersect at most six times, and thus $V_1 \leq 6\binom{7}{2} = 126$, which provides a contradiction. □

Theorem 3.2. *There is no Venn diagram of seven triangles.*

Proof. By Theorem 2.3 and Lemma 3.1. □

4 7-Venn Diagrams of Quadrilaterals

The proof of the previous section raises the question of how close we can get: is there a Venn diagram constructed of seven 4-gons? In this section we answer this question in the affirmative with a simple diagram; this shows that the bound in Theorem 2.3 is tight for $n \leq 7$.

Figure 4 shows a simple 7-Venn diagram of quadrilaterals. The diagram is also *symmetric*: it possesses a rotational symmetry about a centre point, and the seven quadrilaterals are congruent as each maps onto the next by a rotation of $2\pi/7$. The figure is in fact isomorphic as a plane graph (*i.e* can be transformed by continuous deformation in the plane) to the 7-Venn diagram "Victoria" reported in [12, "Symmetric Venn Diagrams"] and [6].

Table 2 gives the coordinates for the four corners of one of the component quadrilaterals: the other six quadrilaterals may be constructed by rotating the given coordinates around the origin by an angle of $2\pi i/n$, for $1 \leq i \leq 6$.

This diagram was discovered using a software tool to manipulate polygons in the plane and compute intersections between them.

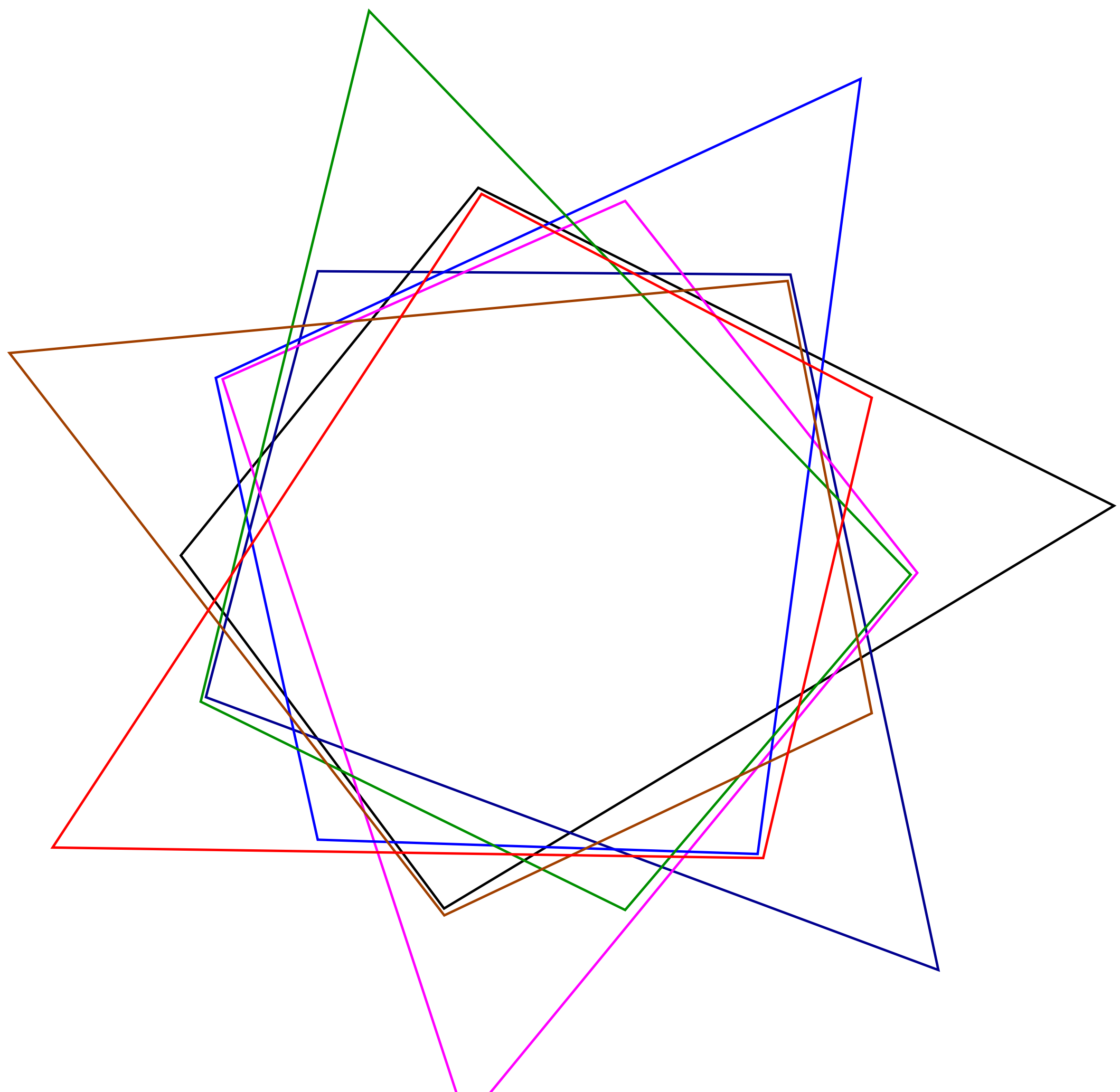

Figure 4: A (symmetric, simple) Venn diagram of seven quadrilaterals.

(x, y)	
$(-0.446,$	$0.000)$
$(-0.123,$	$-0.433)$
$(\ 0.699,$	$0.061)$
$(-0.081,$	$0.451)$

Table 2: Coordinates of corners of a quadrilateral in Figure 4.

5 Open Problems

It is not known whether the bound in Theorem 2.3 is tight for $n > 7$. Note that the non-simple result in Lemma 3.1 works because of the fortuitous fact that $2^n - 2 = \binom{n}{2}2k$ for $n = 7$ and $k = 3$, which is not true for $n \geq 8$. Thus this technique will not work for establishing a non-simple lower bound for $n \geq 8$. A nice open problem is thus to find a tight lower bound on k for the existence of simple and non-simple n-Venn diagrams of k-gons in general. It appears difficult to generalize Theorem 2.3 to the non-simple case; nevertheless we offer:

Conjecture 5.1. *The bound in Theorem 2.3 also holds for non-simple diagrams.*

A *monotone* Venn diagram is one in which every r-region is adjacent to an $(r-1)$-region and an $(r+1)$-region, for $0 < r < n$. It is proven in [3] that a Venn diagram is drawable with convex polygons if and only if it is monotone; specifically, given a simple monotone n-Venn diagram their construction gives a simple diagram of polygons with at most $2^{n+1} - 4$ sides. Not all 7-Venn diagrams can be drawn with quadrilaterals; for example, in the diagram M4 from [12, “Symmetric Venn Diagrams”], each curve has another curve intersect with it 10 times, implying that at least 5-gons are required to draw the figure with k-gons. Thus, what is the maximum over all n-Venn diagrams of the minimum k required to draw each diagram as a collection of k-gons?

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