Symmetric Monotone Venn Diagrams with Seven Curves

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Abstract. An n-Venn diagram consists of n curves drawn in the plane in such a way that each of the 2^n possible intersections of the interiors and exteriors of the curves forms a connected non-empty region. A k-region in a diagram is a region that is in the interior of precisely k curves. A n-Venn diagram is *symmetric* if it has a point of rotation about which rotations of the plane by $2\pi/n$ radians leaves the diagram fixed; it is *polar symmetric* if it is symmetric and its stereographic projection about the infinite outer face is isomorphic to the projection about the innermost face. A Venn diagram is *monotone* if every k-region is adjacent to both some $(k-1)$-region (if $k > 0$) and also to some $k+1$ region (if $k < n$). A Venn diagram is *simple* if at most two curves intersect at any point. We prove that the "Grünbaum " encoding uniquely identifies monotone simple symmetric n-Venn diagrams and describe an algorithm that produces an exhaustive list of all of the monotone simple symmetric n-Venn diagrams. There are exactly 23 simple monotone symmetric 7-Venn diagrams, of which 6 are polar symmetric.

Key words: Venn diagram, symmetry

1 Introduction

1.1 Historical Remarks

The familiar three circle Venn diagram is usually drawn with a three-fold rotational symmetry and the question naturally arises as to whether there are other Venn diagrams with rotational symmetry. Grünbaum [5] discovered a rotationally symmetric 5-Venn diagram. Henderson [7] proved that if an n-curve Venn diagram has an n-fold rotational symmetry then n must be prime. Recently, Wagon and Webb [11] cleared up some details of Henderson's argument. The necessary condition that n be prime was shown to be sufficient by Griggs, Killian and Savage [4] and an overview of these results was given by Ruskey, Savage, and Wagon [10].

A Venn diagram is *simple* if at most two curves intersect at any point. There is one simple symmetric 3-Venn diagram and one simple symmetric 5-Venn diagram. Edwards wrote a program to exhaustively search for polar symmetric

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7-Venn diagrams and he discovered 5 of them, but somehow overlooked a 6-th [3]. His search was in fact restricted to monotone Venn diagrams, which are those that can be drawn with convex curves [1].

A program was written to search for monotone simple symmetric 7-Venn diagrams and 23 of them were reported in the original version of the "Survey of Venn Diagrams" (Ruskey and Weston [9]) from 1997, but no description of the method was ever published and the isomorphism check was unjustified. Later Cao [2] checked those numbers, and provided a proof of the isomorphism check, but again no paper was ever published. In this paper, we justify the isomorphism check and yet again recompute the number of symmetric simple 7-Venn diagrams, using a modified version of the algorithm in [2].

1.2 Definitions

Let $\mathcal{C} = \{C_0, C_1, \ldots, C_{n-1}\}$ be a collection of n finitely intersecting simple closed Jordan curves in the plane. The collection $\mathcal{C}$ is said to be an *n-Venn diagram* if there are exactly 2^n nonempty and connected regions of the form $X_0 \cap X_1 \cap \cdots \cap X_{n-1}$ determined by the n curves in $\mathcal{C}$, where X_i is either the unbounded open exterior or open bounded interior of the curve C_i. Each connected region corresponds to a subset $S \subseteq \{0, 1, \ldots, n-1\}$. A region enclosed by exactly k curves is referred as a *k-region* or a k-set.

A *simple* Venn diagram is one in which exactly two curves cross each other at any point of intersection. In this paper we only consider simple diagrams. A Venn diagram is called *monotone* if every k-region $(0 < k < n)$ is adjacent to both a $(k-1)$-region and a $(k+1)$-region. It is known that a Venn diagram is monotone if and only if it is isomorphic to some diagram in which all of the curves are convex [1].

A Venn diagram is *rotationally symmetric* (usually shortened to *symmetric*) if there is a fixed point p in the plane such that each curve C_i, for $0 \le i < n$, is obtained from C_0 by a rotation of $2\pi i/n$ about p. There is also a second type of symmetry for diagrams drawn in the plane. Consider a rotationally symmetric Venn diagram as being projected stereographically onto a sphere with the south pole tangent to the plane at the point of symmetry p. The projection of the diagram back onto the parallel plane tangent to the opposite pole is called a *polar flip*. If the polar flip results in an isomorphic diagram then the diagram is *polar symmetric*. Figure 1 shows a 7-set polar symmetric Venn diagram (this diagram is known as "Victoria" [3]). In conceptualizing polar flips the reader may find it useful to think of the symmetric diagram as being projected on a cylinder, with the region that intersects all of the sets at the bottom of the cylinder and the empty region at the top of the cylinder. Then the polar flip is akin to turning the cylinder upside-down (see Figure 5).

Two Venn diagrams are generally said to be *isomorphic* if one of them can be changed into the other *or its mirror image* by a continuous transformation of the plane. However, when discussing rotationally symmetric diagrams we broaden this definition to allow for *polar flips* as well. Thus the underlying group of potential symmetries has order $4n$.

As was pointed out earlier, if an n-Venn diagram is symmetric then n is prime. Simple symmetric diagrams for $n = 2, 3, 5, 7$ have been found. The main purpose of this paper is to determine the total number of simple monotone symmetric 7-Venn diagrams. A nice poster of the set of resulting diagrams may be obtained at `http://www.cs.uvic.ca/~ruskey/Publications/Venn7/Venn7.html`.

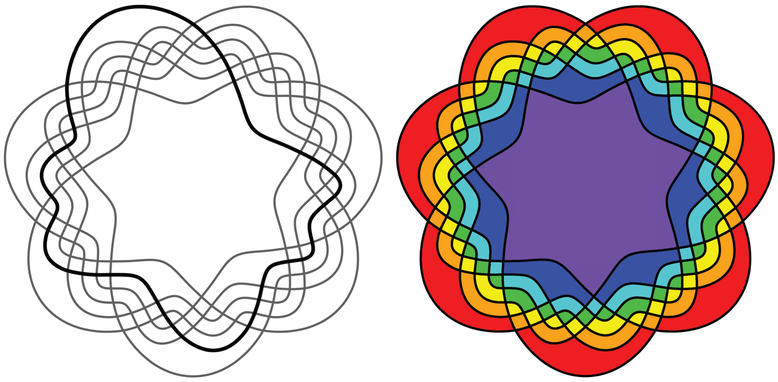

Fig. 1. "Victoria": a simple monotone polar symmetric 7-Venn diagram.

The paper is organized as follows. In Section 2 we outline the classical combinatorial embedding of planar graphs, which is our basic data structure for storing the dual graphs of Venn diagrams. In Section 3 we discuss the representation of Venn diagrams as strings of integers, focussing on those which were used by Grünbaum to manually check whether purported Venn diagrams were Venn diagrams or not, and, if so, whether they were isomorphic.

2 2-Cell Embedding

In this section we outline some of the theory that is necessary for the combinatorial embedding of Venn diagrams in the following sections.

Given a graph G and a surface S, a drawing of G on the surface without edge crossing is called an embedding of G in S. The embedding is 2-cell, if every region of G is homomorphic to an open disk. For a 2-cell embedding of a connected graph with n vertices, m edges and r regions in an orientable surface S_h with h handles we have Euler's formula $n - m + r = 2 - 2h$.

Let $G = (V, E)$ be a finite connected (multi)graph with $V = \{v_1, v_2, \cdots, v_n\}$. For each edge $e \in E$, we denote the oriented edge from v_i to v_j by $(v_i, v_j)_e$ and the opposite direction by $(v_j, v_i)_e$. For each vertex v_i, let E_i be the set of edges oriented from v_i; i.e., $E_i = \{(v_i, v_j)_e : e \in E \text{ for some } v_j \in V\}$. Let Φ_i be the set of cyclic permutations of E_i. The following theorem proved in [12] shows that

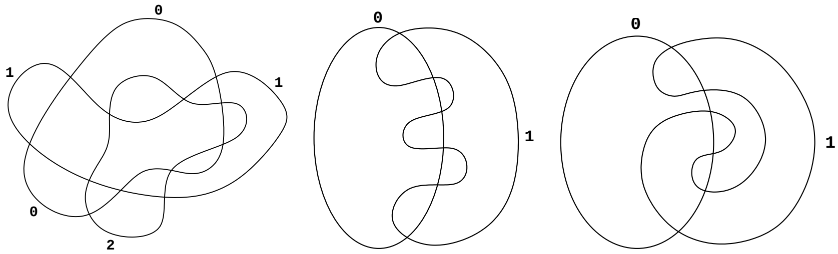

Fig. 2. Examples of normal and non-normal families of intersecting closed curves: (a) is a NFISC, (b) and (c) are not.

there is a one to one correspondence between the set of 2-cell embedding of G and the Cartesian product $\prod \Phi_i$.

Theorem 1. *Let $G = (V, E)$ be a finite connected (multi)graph. Define E_i and Φ_i as above. Then each choice of permutations $(\phi_1, \phi_2, \ldots, \phi_n)$ of $\Phi_1 \times \Phi_2 \times \cdots \times \Phi_n$ determines a 2-cell embedding of G in some orientable surface S_h. Conversely, for any 2-cell embedding of G in S_h, there is a corresponding set of permutations that yields that embedding.*

3 Representations of Symmetric Monotone Venn Diagrams

3.1 G-encoding

A *family of intersecting simple closed* curves (or a FISC) is a collection of simple closed curves enclosing a common non-empty open region and such that every two curves intersect in finitely many points [1].

Definition 1. *A* normal FISC *(or* NFISC*) is a FISC satisfying the following additional conditions:*

- *Every curve touches the infinite face,*
- *The collection is simple, i.e., exactly two curves meet at every point of intersection and they cross each other (each intersection is transverse).*
- *The collection is convex drawable; i.e., it can be transformed into a FISC with all curves convex by a homeomorphic transformation of the plane.*

Let C be an NFISC consisting of n Jordan curves and call the diagram consisted of these n curves an *n-diagram*. Choosing an arbitrary curve as curve 0, we label all n curves by their clockwise appearance on the outmost region. Let M be the number of times the curves touch the infinity face, $M \geq n$. A *G-encoding* consists of $M+1$ sequences and an $M \times n$ matrix F. The first sequence,

(a)

i	I_i	w_i
0	0	1 2 2 2 1 2 1 1
1	1	2 0 2 0 0 2 2 0
2	2	0 1 1 1 0 0 0 1
3	0	1 1 1 2 2 2 1 2
4	1	0 2 2 0 2 0 2 0
		0 1 2 3 4 5 6 7

(b)

	0	1	2
0	∞	4	5
1	7	∞	2
2	4	7	∞
3	∞	6	7
4	3	∞	6

Fig. 3. (a) G-encoding of Figure 2(a). (b) The corresponding F-matrix.

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call it $I = I_0, I_1, \ldots, I_{M-1}$ has length M. Starting with curve 0, it specifies the curves encountered as we walk around the outer face of the n-diagram in clockwise direction. Thus, $I_i \in \{0, 1, \ldots, n-1\}$. Each element of I corresponds to a curve segment in the outer face of the diagram. For each curve c, the *first segment* is the one which corresponds to the first appearance of c in I. The other M sequences are denoted $w_0, w_1, \ldots, w_{M-1}$. Sequence w_i records intersections along curve I_i as a sequence of integers, indicating the curves encountered at intersection points. As usual, the curves are traversed in a clock-wise order.

Among all intersections of the traversal starting at I_i with curve j, let $F[i,j]$ be the index of the first intersection with curve j after curve j touches the outer face for the first time. That is, if $p_1, p_2, \ldots, p_t$ are the indices of the intersections with curve j in sequence w_i, the first segment of curve j will eventually hit the outer face, say between intersections at positions p_{s-1} and p_s; then $F[i,j] = p_s$.

Figure 3 shows the G-encoding of the 3-diagram of Figure 2(a). The first table shows the I and w_i sequences. The second table is the F matrix. Since the curves are not self intersecting, we define $F[i,j] = \infty$ if $j = I_i$. It is worth noting that in general the w_i sequences may have different lengths.

By constructing a circular list of oriented edges for each vertex (point of intersection), it can be shown that there is a correspondence between a 2-cell embedding of an NFISC n-diagram and its G-encoding.

Theorem 2. *Each G-encoding of an NFISC of n Jordan curves uniquely determines a 2-cell embedding of the n-diagram in some sphere S_0.*

Note that for a non-NFISC, the G-encoding does not necessarily determine a unique diagram. For example the two non-NFISC diagrams (b) and (c) in Figure 2 have the same G-encoding.

3.2 The Grünbaum Encoding

Grünbaum encodings were introduced by Grünbaum as a way of hand-checking whether two Venn diagrams are distinct. However, no proof of correctness of the method was ever published. The Grünbaum encoding of a simple symmetric monotone Venn diagram consists of four n-ary strings, call them w, x, y, z.

String w is obtained by first labeling the curves from 0 to $n-1$ according to their clockwise appearance on the outer face and then following curve 0 in a clockwise direction, starting at a point where it touches the outermost region and meets curve 1, recording its intersections with the other curves, until we reach again the starting point.

String x is obtained by first labeling the curves in the inner face starting at 0 in a clockwise direction and then by following curve 0 in a clockwise direction starting at the intersection with curve 1.

Strings y and z are obtained in a similar way but in a counter-clockwise direction. First, curves are re-labeled counter-clockwise as they appear on the outer face. Then strings y and z are obtained by following curve 0 in a counter-clockwise direction starting from the outermost and innermost regions respectively and recording its intersection with other curves. The Grünbaum encoding of the Venn diagram shown in Figure 1 is given below.

w: 1 4 2 5 3 6 1 6 3 5 3 6 2 5 1 6 1 5 3 6 2 5 1 4 2 6 1 6 2 5 1 4 2 4 1 6
x: 1 6 3 5 3 6 2 5 1 6 1 5 3 6 2 5 1 4 2 6 1 6 2 5 1 4 2 4 1 6 1 4 2 5 3 6
y: 1 6 3 5 3 6 2 5 1 6 1 5 3 6 2 5 1 4 2 6 1 6 2 5 1 4 2 4 1 6 1 4 2 5 3 6
z: 1 4 2 5 3 6 1 6 3 5 3 6 2 5 1 6 1 5 3 6 2 5 1 4 2 6 1 6 2 5 1 4 2 4 1 6

Property 1. Each string of the Grünbaum encoding of a simple symmetric monotone n-Venn diagram has length $(2^{n+1}-4)/n$.

Proof. Clearly each string will have the same length, call it L. An n-Venn diagram has 2^n regions, and in a simple diagram every face in the dual is a 4-gon. We can therefore use Euler's relation to conclude that the number of intersections is $2^n - 2$. By rotational symmetry every intersection represented by a number in the encoding corresponds to $n-1$ other intersections. However, every intersection is represented twice in this manner. Thus $nL = 2(2^n-2)$, and hence $L = (2^{n+1}-4)/n$. □

According to the definition of Grünbaum encoding, each string starts with 1 and ends with $n-1$. Given string w of the Grünbaum encoding, we can compute the other three strings. Let $L = (2^{n+1}-4)/n$ denote the length of the Grünbaum encoding, and let $w[i]$, $x[i]$, $y[i]$ and $z[i]$ be the ith element of w, x, y and z, respectively, where $0 \le i \le L-1$. Then, clearly,

$$y[i] = n - w[L-i-1] \quad \textit{and} \quad z[i] = n - x[L-i-1].$$

To obtain x, we first find out the unique location in w where all curves have been encountered an odd number of times (and thus we are now on the inner face), then shift w circularly at this location. The string z can be easily inferred from y in a similar manner.

Three isomorphic Venn diagrams may be obtained from any Venn diagram by "flipping" and/or "polar flipping" mappings. The strings x, y and z are the first strings of the Grünbaum encodings of these isomorphic diagrams. So we can easily verify isomorphisms of any Venn diagram using the Grünbaum encoding. Due to space limitations the proof of the following theorem is omitted.

Theorem 3. *Each Grünbaum encoding determines a unique simple symmetric monotone n-Venn diagram (up to isomorphism).*

Using Grünbaum encoding of a Venn diagram, we can also verify whether it is polar symmetric or not by the following theorem.

Theorem 4. *An n-Venn diagram is polar symmetric if and only if the two string pairs (w, z) and (x, y) of its Grünbaum encoding are identical.*

Proof. For a given Venn diagram D with Grünbaum encoding (w, x, y, z) there are three isomorphic Venn diagram obtained by horizontal, vertical and polar flips with Grünbaum encodings (y, z, w, x), (x, w, z, y) and (z, y, x, w) respectively. Let D' denotes the Venn diagram obtained by polar flip mapping of D. If D is polar symmetric, then it remains invariant under polar flips So D and D' must have the same Grünbaum encoding, that is, $(w, x, y, z) = (z, y, x, w)$. Therefore, for a polar symmetric Venn diagram we have $w = z$ and $x = y$.

Conversely, suppose we have a Venn diagram D with Grünbaum encoding (w, x, y, z) such that $w = z$ and $x = y$. Then $(w, x, y, z) = (z, y, x, w)$. So the isomorphic Venn diagram D' obtained by polar flip mapping of D, has the same Grünbaum encoding as D. So by theorem 3 D and D' are equivalent and the diagram is polar symmetric. □

3.3 The Matrix Representations of Monotone Diagrams

Because of the property of symmetry, an n-Venn symmetric diagram may be partitioned into n identical sectors. Each sector is a pie-slice of the diagram between two rays from the point of symmetry offset by $2\pi/n$ radians from each other. So the representation of one sector is sufficient to generate the whole diagram.

Given a sector of a simple monotone n-Venn diagram, one can map it to a graph consisting of n intersecting polygonal curves (which we call *polylines*), as shown in Figure 4. Putting 0s between these n polylines and 1s at the intersections gives us a 0/1 matrix. We then can expand the matrix by appending identical matrix blocks to generate a matrix that represent the whole Venn diagram.

An n-Venn diagram has exactly 2^n regions. Among them one is most inside (inside all curves) and one is most outside (outside all curves). The rest of $2^n - 2$ regions are evenly distributed in each sector. Hence in each sector there are $(2^n - 2)/n$ regions. We use a 1 to indicate the starting point of a region and the ending point of the adjacent region. This implies that there are exactly $(2^n - 2)/n$ 1's in the matrix. So one can always use a $(n - 1) \times (2^n - 2)/n$ 0/1 matrix to represent one sector and use a $(n - 1) \times (2^n - 2)$ 0/1 matrix to represent the whole diagram.

If a matrix (a_{ij}), $i = 0, 1, \cdots, n - 2$, $j = 0, 1, \cdots, (2^n - 2)/n - 1$, is a representation of a Venn diagram, then any matrix obtained by a shift of some number of columns is also a representation of the same diagram. Therefore we

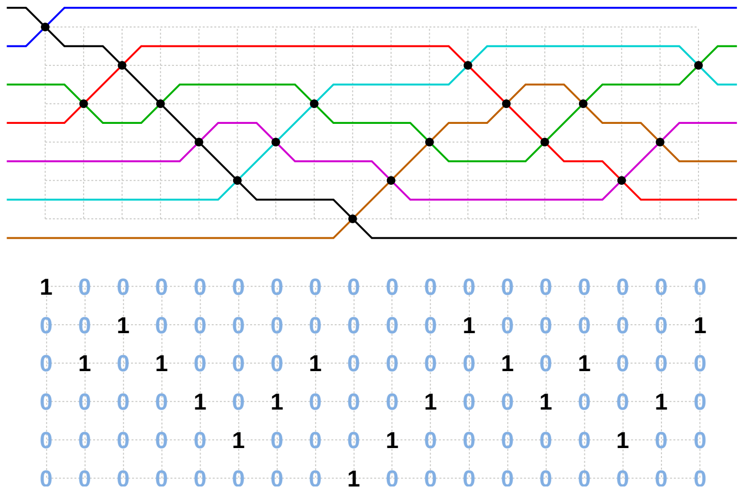

Fig. 4. Matrix representation of Victoria

can always shift the representation matrix so that $a_{00} = 1$. The matrix with 1 at the first entry is called the *standard* representation matrix.

The matrix representation of a simple symmetric monotone Venn diagram of n curves has the following properties:

(a) The total number of 1's in the matrix is $(2^n - 2)/n$, with one 1 in each column.
(b) There are $\binom{n}{k}/n$ 1s in the kth row, for $k = 1, 2, \ldots, n-1$.
(c) There are no two adjacent 1's in the matrix.

Note that different 0/1 matrices could represent isomorphic Venn diagrams. How do we know whether a given 0/1 matrix represents a "new" Venn diagram? The Grünbaum encoding provides a convenient way to solve this problem.

4 The Algorithm

The algorithm to find all symmetric monotone Venn diagrams consists of the following four steps.

1. **Step one:** Generate all possible standard 0/1 matrices with $n-1$ rows and $(2^n - 2)/n$ columns that satisfy (a), (b) and (c). To generate each row we are generating restricted combinations; e.g., all bitstrings of length 18 with k 1s, no two of which are adjacent.
2. **Step two:** Check validity. For each matrix V generated in step one, by appending it $n-1$ times, we first extend the matrix to a matrix X that represents the whole potential Venn diagram. A valid matrix must represent exactly $2^n - 2$ distinct regions of the corresponding Venn diagram. The two

other regions are the outermost and the innermost regions. Each region is specified by its rank defined as

$$\text{rank } = 2^0 x_0 + 2^1 x_1 + \cdots + 2^{n-1} x_{n-1},$$

where $x_i = 1$ if the curve i is outside of the region and $x_i = 0$ otherwise.
In order to check the regions and generate the Grünbaum encoding, an $n \times (2^n - 2)$ matrix C called the P-matrix is generated. The P-matrix gives us another which represents the curves of the Venn diagram (see Table 1). The first column of C is set to $[0, 1, \ldots, n-1]^T$ and for each successive column j, $1 \leq j < 2^n - 2$, we use the same entries of column $j-1$ and then swap C_{ij}, $C_{(i+1)j}$ if $X_{i(j-1)} = 1$.
To check the validity of matrix X, we scan it column by column from left to right. Each 1 indicates the end of one region and start of another region. The entries in the same column of matrix C are used to compute the rank of the regions. The generated matrix is a valid representation of a Venn diagram if $2^n - 2$ distinct regions are found by scanning the whole matrix, which will only occur if each of the rank calculations are different.

0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	0	3	3	3	3	3	3	3	3	3	5	5	5	5	5	5
2	2	3	0	2	2	2	2	5	5	5	5	3	6	6	2	2	2
3	3	2	2	0	4	4	5	2	2	2	6	6	3	2	6	6	4
4	4	4	4	4	0	5	4	4	4	6	2	2	2	3	3	4	6
5	5	5	5	5	5	0	0	0	6	4	4	4	4	4	4	3	3
6	6	6	6	6	6	6	6	6	0	0	0	0	0	0	0	0	0

Table 1. The first 18 columns of the P-matrix of Victoria.

3. **Step three:** Generate the Grünbaum encoding. To generate Grünbaum codes, we first relabel the polylines by the order of appearances in the first row so that they are labeled with 0, 1, ..., $n-1$ (for Table 1 the relabeling permutation is 0124536). Following polyline 0 and recording its intersections with the other polylines, we have the first string w of the Grünbaum encoding. The other three strings, x, y and z, are computed from w.
4. **Step four:** Eliminate isomorphic solutions. By sorting the four strings of the Grünbaum encoding of each produced Venn diagram into lexicographic order and comparing them with the encodings of previously generated Venn diagrams, we eliminate all isomorphic solutions. If the current diagram is not isomorphic to any of previously discovered diagrams, then it will be added to the solution set.

Checking all possible 0/1 matrices for $n = 7$, we found exactly 23 non-isomorphic symmetric monotone Venn diagrams, of which 6 diagrams are polar symmetric. See Figures 6 and 7.

5 Drawing

The polyline diagram in figure 4 shows one sector of the cylindrical projection of Victoria. So given the matrix representation of a Venn diagram, one can easily get its cylindrical projection by computing the cylindrical coordinates of each intersection point. Because of the symmetry, it is sufficient to compute the coordinates only for the first curve. We also need extra points to specify peeks and valleys. To get a visually pleasing shape, we moved the points in such a way that at each point the line segments are perpendicular to each other. Figure 5 shows the resulting representation for Victoria.

The Cartesian coordinates of each point on the plane can be obtained from its cylindrical coordinates. Then we draw the first curve by applying spline interpolation to the computed coordinates. The other six curves are simply drawn by rotating the first curve about the point of symmetry. Figures 7 and 6 show drawings of all 23 diagrams, as constructed by this method.

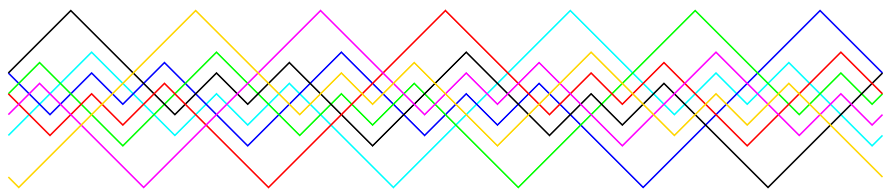

Fig. 5. Cylindrical representation of Victoria

6 Conclusions and open problems

A matrix representation of simple symmetric monotone Venn diagrams has been introduced. We proved that Grünbaum encoding can be used to check the isomorphism and polar symmetry of simple symmetric monotone Venn diagrams. Using an exhaustive search algorithm we verified that there are exactly 23 non-isomorphic simple symmetric monotone 7-Venn diagrams, which 6 of them are polar symmetric. Below is the list of some related open problems: (a) Find the total number of simple symmetric *non-monotone* 7-Venn diagrams. (b) Is there a simple symmetric Venn diagram for $n = 11$?

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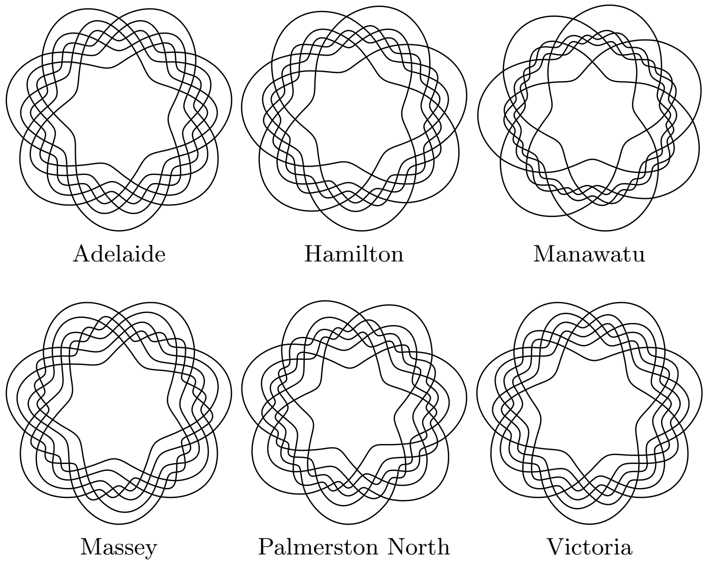

Fig. 6. All simple monotone polar symmetric 7-Venn diagrams

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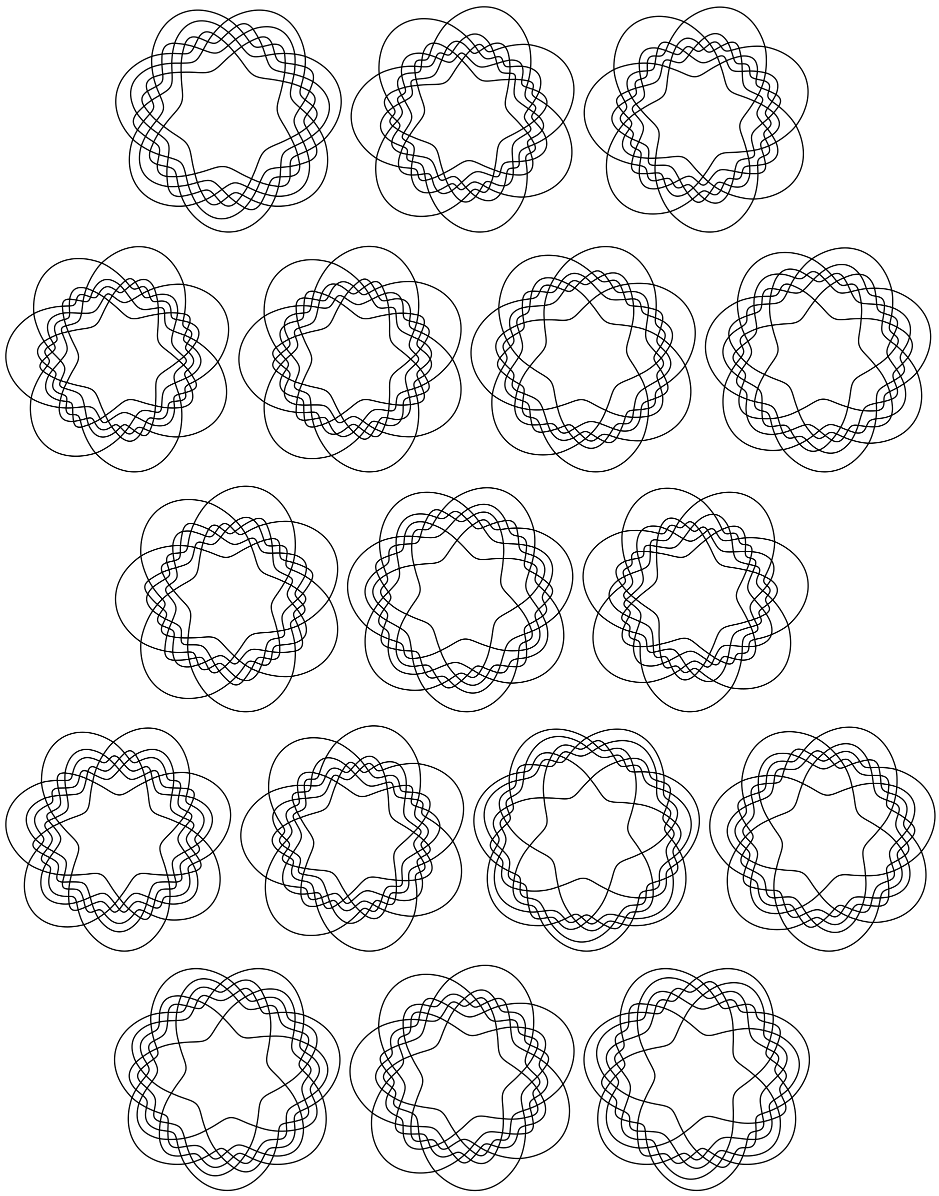

Fig. 7. All 17 simple monotone non-polar symmetric 7-Venn diagrams