

NESTED RECURRENCE RELATIONS WITH CONOLLY-LIKE SOLUTIONS

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ABSTRACT. A non-decreasing sequence of positive integers is (α, β) -Conolly, or *Conolly-like* for short, if for every positive integer m the number of times that m occurs in the sequence is $\alpha + \beta r_m$, where r_m is 1 plus the 2-adic valuation of m . A recurrence relation is (α, β) -Conolly if it has an (α, β) -Conolly solution sequence. We discover that Conolly-like sequences often appear as solutions to nested (or meta-Fibonacci) recurrence relations of the form $A(n) = \sum_{i=1}^k A(n - s_i - \sum_{j=1}^{p_i} A(n - a_{ij}))$ with appropriate initial conditions. For any fixed integers k and p_1, p_2, \dots, p_k we prove that there are only finitely many pairs (α, β) for which $A(n)$ can be (α, β) -Conolly. For the case where $\alpha = 0$ and $\beta = 1$, we provide a bijective proof using labelled infinite trees to show that, in addition to the original Conolly recurrence, the recurrence $H(n) = H(n - H(n - 2)) + H(n - 3 - H(n - 5))$ also has the Conolly sequence as a solution. When $k = 2$ and $p_1 = p_2$, we construct an example of an (α, β) -Conolly recursion for every possible (α, β) pair, thereby providing the first examples of nested recursions with $p_i > 1$ whose solutions are completely understood. Finally, in the case where $k = 2$ and $p_1 = p_2$, we provide an if and only if condition for a given nested recurrence $A(n)$ to be $(\alpha, 0)$ -Conolly by proving a very general ceiling function identity.

1. INTRODUCTION

In this paper we analyze recurrence relations of the form

$$(1.1) \quad R(n) = \sum_{i=1}^k R \left(n - s_i - \sum_{j=1}^{p_i} R(n - a_{ij}) \right)$$

where the parameters k , s_i , p_i and a_{ij} are positive integer constants (with the exception of s_i , which can be equal to 0). We assume c initial conditions $R(1) = \xi_1, R(2) = \xi_2, \dots, R(c) = \xi_c$, where unless otherwise indicated, $\xi_i > 0$. Equation (1.1) has a well-defined solution sequence as long as for each i in $\{1, 2, \dots, k\}$, the argument $n - s_i - \sum_{j=1}^{p_i} R(n - a_{ij})$ is positive.

We use the following notation for Equation (1.1) and optionally its initial conditions,

$$(1.2) \quad \langle s_1; a_{11}, a_{12}, \dots, a_{1p_1} : s_2; a_{21}, a_{22}, \dots, a_{2p_2} : \dots : s_k; a_{k1}, a_{k2}, \dots, a_{kp_k} \rangle [\xi_1, \xi_2, \dots, \xi_c].$$

For convenience we use this notation to refer to both the recursion and its solution sequence even in the situation where we are uncertain that a solution exists. For example Hofstadter's Q -sequence ([9]), defined by $Q(1) = Q(2) = 1$ and $Q(n) = Q(n - Q(n - 1)) + Q(n - Q(n - 2))$ for $n > 2$ is written $Q = \langle 0; 1 : 0; 2 \rangle [1, 1]$; Q is a famous example where it is not known whether or not a solution exists.

We classify recurrences of the form (1.1) according to the values of the parameters p_i and k . We say the above recursion has *order* $\mathbf{p} = (p_1, p_2, \dots, p_k)$, and for convenience we say it

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has order p if $p_1 = p_2 = \cdots = p_k = p$. We refer to k as the *arity* of a recursion. The case $k = 1$ is uninteresting and yields only easily-understood periodic sequences (see [7] for an outline of the required argument in a special case), so we only consider recursions of arity at least 2.

To date, recursions of the form (1.1) have been examined only for $p = 1$ (see, for example, [1, 5, 8, 11, 14, 15]). As with nested recursions of order 1, no general solution methodology is available for general order p . Therefore a natural starting point for our analysis is the gathering and organization of experimental data in a variety of situations.

Our empirical investigations focused on $k = 2$ and $p > 1$, ie, recursions of the form

$$(1.3) \quad R(n) = R(n - s - R(n - a_1) \cdots - R(n - a_p)) + R(n - t - R(n - b_1) \cdots - R(n - b_p)).$$

When it is convenient, we assume without loss of generality that $0 \leq s \leq t$ and $1 \leq a_1 \leq \cdots \leq a_p$, and $1 \leq b_1 \leq \cdots \leq b_p$. Using the notation introduced in (1.2), we write $R = \langle s; a_1, \dots, a_p : t; b_1, \dots, b_p \rangle$.

Our empirical work indicated that many combinations of parameters and initial conditions yield a solution sequence reminiscent of the well-known Conolly sequence [6]. This sequence is the solution to the *nested* (also called self-referencing or meta-Fibonacci) recurrence relation $C(n) = \langle 0; 1 : 1; 2 \rangle[1, 2]$. We say it is *slowly growing* or *slow* because it has the property that successive differences are either 0 or 1. Any slowly growing sequence can be described by its *frequency sequence*, $\phi_C(m)$, that counts the number of times that m occurs in $C(n)$. It is shown in [11] that $\phi_C(m)$ equals r_m , the so-called ‘‘ruler function’’. The ruler function r_m is defined as one plus the 2-adic valuation of m (the exponent of 2 in the prime factorization of m). Thus $r_m = 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, \dots$

We observed that the aforementioned solution sequences related to the Conolly sequence have frequency sequences of the form $\alpha + \beta r_m$, for integers α and β ; we call such sequences *Conolly-like*. When we want to specify α and β , we say the sequence is the (α, β) -*Conolly* sequence. Correspondingly, we say that a recurrence relation is Conolly-like if it has a Conolly-like solution sequence and is (α, β) -*Conolly* when the solution is (α, β) -Conolly.¹

Because of their close connection to the ruler function, Conolly-like sequences have many useful properties. In particular, following the approach in [11, 12] it is straightforward to show that $\frac{z}{1-z} \prod_{n \geq 0} \left(1 + z^{2^n \alpha + (2^{n+1} - 1)\beta}\right)$ is the generating function for the general (α, β) -Conolly sequence². Thus, proving that a given recursion is (α, β) -Conolly also determines the generating function for its Conolly-like solution.

In the balance of this paper we focus on proving the recurrences of the form (1.3) that we have identified experimentally are indeed Conolly-like. To do so we develop a proof technique for general p that adapts the tree-based combinatorial interpretations invented in [11, 10, 12] for the case $p = 1$. Our idea is to build upon these combinatorial interpretations, which identify an infinite tree associated with the Conolly recursion and the recursion

¹Note that a recurrence is Conolly-like if there is at least one set of initial conditions for which its solution is Conolly-like; in general, for an arbitrary set of initial conditions, the solution, if it exists, will not be Conolly-like.

²The central idea of this technique is to find a recursive description of the successive differences of the (α, β) -Conolly sequence (these differences are either 0 or 1 because it is slowly growing), and then find the generating function of these differences.

$\langle 0; 1 : 2; 3 \rangle[1, 1, 2]$, respectively. These order 1 recursions have solutions with frequency sequences r_m and 2, respectively. For higher order Conolly-like recursences our rough idea is to create new trees as “linear combinations” of the original two trees.

Before we can apply the above approach effectively we must narrow the range of possible recursions of the form (1.3) that are candidates for having Conolly-like solutions. We do so in Section 2 where we derive asymptotic properties of any solution to (1.1). We compare these properties with the known asymptotic properties of an (α, β) -Conolly sequence in order to identify, for any given order p , necessary restrictions on the parameters α and β . These findings provide important guidance for the empirical search procedures that we adopt and motivate many of the results in the remainder of the paper.

Since our primary focus is on Conolly-like sequences, it is natural to ask if the Conolly recursion $\langle 0; 1 : 1; 2 \rangle[1, 2]$ is the only recursion of the form (1.3) whose solution is the Conolly sequence. From the results in Section 2 we know that any such recursion must be order 1. In Section 3 we use the tree technique outlined above to prove that $\langle 0; 2 : 3; 5 \rangle[1, 2, 2, 3, 4]$ is another order 1 recursion whose solution is the Conolly sequence. Based on our experimental results we conjecture that there are no other 2-ary order 1 $(0, 1)$ -Conolly recursions. Further, we completely describe all 2-ary order 1 $(2, 0)$ -Conolly recursions. From Section 2 we know that all Conolly-like solutions to 2-ary recursions of order 1 are either $(2, 0)$ -Conolly or $(0, 1)$ -Conolly, giving us a near-complete description.

In Section 4, we apply a similar tree technique, given any p, α, β satisfying the necessary condition in Section 2, to construct an order p (α, β) -Conolly recursion of the form (1.3). This proves that these conditions are also sufficient. We conclude this section by constructing explicitly, given any $m > 0$ and an (α, β) -Conolly recursion with order p , a related $(m\alpha, m\beta)$ -Conolly recursion with order mp .

Up to this point we have focused on finding examples of (α, β) -Conolly recursions. In Section 5 we turn to the following question: for a given (α, β) , what is the complete list of (α, β) -Conolly recursions of the form (1.3)? For $\beta = 0$ and arbitrary order p we provide a complete answer. Further, for $p = 2$ we conjecture a complete list of Conolly-like recursions. We conclude in Section 6 with a discussion of future directions for further work.

2. NARROWING THE SEARCH FOR CONOLLY-LIKE RECURRENCES

Equation (1.3) describes a very large class of recursions. Furthermore, it is not clear for which, if any, of the infinitely many pairs (α, β) a given recursion might be (α, β) -Conolly. However, specifying that a sequence is (α, β) -Conolly implies very strong conditions on its asymptotic behavior determined by α and β . At the same time any sequence that solves (1.3) must have asymptotic properties that depend upon the value of p . Therefore a Conolly-like solution to (1.3) must satisfy both sets of requirements, necessitating a relationship between p and (α, β) . In what follows we derive this relationship; in doing so, because it is no more difficult, we establish a more general result that applies to all recursions (1.1).

The first step is to determine, given a limit L , what parameters for (1.1) can lead to a solution $A(n)$ such that the ratio $A(n)/n$ converges to L .

Theorem 2.1. *Let $A(n)$ be the solution sequence of a meta-Fibonacci recursion of the form (1.1). Suppose that the ratio $A(n)/n$ has a nonzero limit L . Then $L = \frac{k-1}{\sum_{i=1}^k p_i}$. If $p_i = p$ for all i and $k = 2$, which is the focus of our attention in this paper, then $L = \frac{1}{2p}$.*

Note that the hypothesis $L > 0$ in Theorem 2.1 rules out the trivial zero sequence solution.

Proof. Let $A(n)$ satisfy the recursion $A(n) = \sum_{i=1}^k A\left(n - s_i - \sum_{j=1}^{p_i} A(n - a_{ij})\right)$. As a first step, we will show that we cannot have $p_i L > 1$ for any i . Assume to the contrary that for some h , $p_h L > 1$. Then for sufficiently large n , $\sum_{j=1}^{p_h} A(n - a_{hj}) > n - s_h$. But then the argument of the h^{th} summand is negative, which is impossible since $A(n)$ is a solution. Thus, for all i , $p_i L \leq 1$. We will later be able to show that, in fact, $p_i L < 1$ for all i .

Suppose there exists at least one value of i for which $p_i L = 1$. Without loss of generality let $p_1 L = 1$. Since $p_1 \geq 1$, we have $\frac{A(n)}{n} \rightarrow L \leq 1$, so there exists some positive constant λ such that for all n , $A(n) \leq \lambda n$. Then for all n , $A(n - s_1 - \sum_{j=1}^{p_1} A(n - a_{1j})) \leq \lambda(n - s_1 - \sum_{j=1}^{p_1} A(n - a_{1j}))$. It follows that $\frac{A(n - s_1 - \sum_{j=1}^{p_1} A(n - a_{1j}))}{n} \leq \lambda - s_1/n - \lambda \sum_{j=1}^{p_1} \frac{A(n - a_{1j})}{n}$. But $\frac{n}{n - a_{1j}} \rightarrow 1$, so $\frac{A(n - a_{1j})}{n} \rightarrow L$, from which we deduce that $\lim_{n \rightarrow \infty} \frac{A(n - s_1 - \sum_{j=1}^{p_1} A(n - a_{1j}))}{n} \leq \lambda - \lambda \sum_{j=1}^{p_1} L = \lambda - \lambda p_1 L = 0$.

Now, write

$$\frac{A(n)}{n} = \frac{A(n - s_1 - \sum_{j=1}^{p_1} A(n - a_{1j}))}{n} + \frac{\sum_{i=2}^k A\left(n - s_i - \sum_{j=1}^{p_i} A(n - a_{ij})\right)}{n}$$

and take the limit as $n \rightarrow \infty$ on both sides. The first term on the right vanishes. This argument shows that in evaluating L we can ignore any summands with index i such that $p_i L = 1$. Further, it confirms that we cannot have $p_i L = 1$ for all i , since then $L = 0$, which contradicts our assumption.

Combining the above results we may safely assume that $p_i L < 1$ for all i . For each i define $\kappa_i(n) := n - s_i - \sum_{j=1}^{p_i} A(n - a_{ij})$. Then $\lim_{n \rightarrow \infty} \frac{\kappa_i(n)}{n} = 1 - p_i L > 0$, and thus $\lim_{n \rightarrow \infty} \kappa_i(n) = \infty$. But since $\lim_{n \rightarrow \infty} \frac{A(n)}{n} = L$ it follows that for all i ,

$$\frac{A\left(n - s_i - \sum_{j=1}^{p_i} A(n - a_{ij})\right)}{n - s_i - \sum_{j=1}^{p_i} A(n - a_{ij})}$$

converges to L as $n \rightarrow \infty$.

We can write

$$\begin{aligned} \frac{A(n)}{n} &= \sum_{i=1}^k \frac{A\left(n - s_i - \sum_{j=1}^{p_i} A(n - a_{ij})\right)}{n} \\ &= \sum_{i=1}^k \left(\frac{n - s_i - \sum_{j=1}^{p_i} A(n - a_{ij})}{n} \right) \left(\frac{A\left(n - s_i - \sum_{j=1}^{p_i} A(n - a_{ij})\right)}{n - s_i - \sum_{j=1}^{p_i} A(n - a_{ij})} \right). \end{aligned}$$

Taking the limit on both sides as $n \rightarrow \infty$ we get $L = \sum_{i=1}^k (1 - p_i L)(L)$, from which we conclude (since $L > 0$) that $1 = k - L \sum_{i=1}^k p_i$, or $L = \frac{k-1}{\sum_{i=1}^k p_i}$. \square

In Theorem 2.1 we assume *a priori* the existence of the limit L . This assumption is needed, a fact we explain further at the end of this section.

We now apply Theorem 2.1 to Conolly-like meta-Fibonacci sequences.

Corollary 2.1. *Let $A(n)$ be an (α, β) -Conolly sequence satisfying a recursion of the form (1.3). Then $\alpha + 2\beta = 2p$, $\alpha + \beta > 0$, and $\beta \geq 0$.*

Proof. Any slow sequence $A(n)$ has a positive frequency sequence, $\phi_A(m)$. But $r_m = 1$ whenever m is odd, so to have $\phi_A(m) = \alpha + \beta r_m > 0$, we must have $\alpha + \beta > 0$ (the converse obviously also holds). Because r_m is unbounded and α is fixed, we must have $\beta \geq 0$ to ensure that $\phi_A(m) = \alpha + \beta r_m > 0$.

Using the well-known fact that $\lim_{n \rightarrow \infty} (1/n) \sum_{m=1}^n r_m = 2$ (see [2], Section 3.2, p. 74), we have that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n (\alpha + \beta r_m) = \alpha + 2\beta$.

The sequence $A(n)$ is (α, β) -Conolly so has frequency sequence $\alpha + \beta r_m$. Consider the subsequence n_h where n_h is the largest integer with $A(n_h) = h$. Then $n_h = \sum_{m=1}^h (\alpha + \beta r_m)$. So it follows that $\lim_{h \rightarrow \infty} \frac{A(n_h)}{n_h} = \lim_{h \rightarrow \infty} \frac{h}{\sum_{m=1}^h \alpha + \beta r_m} = \frac{1}{\alpha + 2\beta}$. But it can be readily shown that $\frac{A(n)}{n}$ and $\frac{A(n_h)}{n_h}$ are close as $n \rightarrow \infty$ so that both sequences converge to the same limit. Hence $\lim_{n \rightarrow \infty} \frac{A(n)}{n} = \frac{1}{\alpha + 2\beta}$. Thus by Theorem 2.1, $\alpha + 2\beta = 2p$. \square

From Corollary 2.1 it follows that for given order p there are $2p$ possible pairs (α, β) , one for each of $\beta = 0, 1, \dots, 2p - 1$. For $p = 1, 2, 3, 4$ these are given in Table 1; recall that every (α, β) -pair uniquely defines the corresponding Conolly-like sequence.

TABLE 1. Possible (α, β) -pairs for orders 1 to 4

Order	Possible Pairs (α, β)
1	$(2, 0), (0, 1)$
2	$(4, 0), (2, 1), (0, 2), (-2, 3)$
3	$(6, 0), (4, 1), (2, 2), (0, 3), (-2, 4), (-4, 5)$
4	$(8, 0), (6, 1), (4, 2), (2, 3), (0, 4), (-2, 5), (-4, 6), (-6, 7)$

For any fixed p we now have a list of the finitely many (α, β) -Conolly sequences that could solve an order p recurrence. However, so far we have only necessary conditions: for $p > 1$ we haven't yet shown how to characterize which of the recursions (1.3) are (α, β) -Conolly.

Toward this end we wrote a program (in C) to systematically and efficiently test restricted parts of the parameter space. Our search process is best described in the context of order 2, where there are only 4 possible (α, β) pairs, namely $(4, 0)$, $(2, 1)$, $(0, 2)$, and $(-2, 3)$, and 6 parameters: $\langle s; a, b : t; c, d \rangle$.

First we fixed an (α, β) pair. Then we explored parameter sets that satisfied $0 = s \leq t \leq 10$, $1 \leq a \leq b \leq 12$, and $1 \leq c \leq d \leq 30$.³ When $s = t$ we only examined those with pairs (a, b) that are lexicographically less than or equal to (c, d) , so as to avoid duplication.

With each parameter set we supplied a sufficient number of initial conditions in order to seed the recurrence. These values were the first 20 terms of the (α, β) -Conolly sequence. We generated the first 1000 values for each sequence that was well-defined to that point and compared the result to the intended (α, β) -Conolly sequence.

From our experimental data we discovered that for each of the four 2-ary order 2 (α, β) -pairs indicated in Table 1 there appear to be multiple Conolly-like recursions, including the recursion $\langle 0; 1, 3 : \gamma; \gamma + 1, \gamma + 3 \rangle$ where $\gamma = \alpha + \beta$. Based on this experimental evidence we conjectured that an analogous result is true for 2-ary recursions of any order. This is the content of Theorem 4.1 below. Together with Corollary 2.1 above this establishes necessary and sufficient conditions on α and β for the existence of at least one 2-ary order p (α, β) -Conolly recursion.

³Earlier experimental probing suggested that $s > 0$ yielded no Conolly-like recursions.

Based on our experimental evidence, we conjectured that only finitely many k -ary order p (α, β) -Conolly recursions exist for fixed k and p , and $\beta > 0$. For $\beta = 0$ we show in Section 5 that there are infinitely many recursions of the form (1.3); in fact, we demonstrate that for fixed α, k, p_i there are either no $(\alpha, 0)$ -Conolly recursions of the form (1.1) or infinitely many.

We close this section by addressing the issue we raised following Theorem 2.1, namely, that there exist meta-Fibonacci sequences $A(n)$ that solve recursion (1.1) for which $A(n)/n$ does not approach a limit. This demonstrates the necessity of the strict conditions on Theorem 2.1. One known example is Hofstadter's recursion $Q = \langle 0; 1 : 0; 2 \rangle$ with alternate initial conditions 3, 2, 1 [7], the solution to which is the quasi-periodic sequence $Q(3w + 1) = 3, Q(3w + 2) = 3w + 2$ and $Q(3w) = 3w - 2$. The following result illustrates how to construct such sequences for any values of k and p_i . In our approach we “weave” together two different solutions to the same recursion.⁴

Theorem 2.2. (*Fixed Order Interleaving Theorem*) *Let $A(n)$ and $B(n)$ both satisfy the same recursion of the form (1.1) with initial conditions $A(1), \dots, A(r)$ and $B(1), \dots, B(r)$ respectively, so for $n > r$,*

$$A(n) = \sum_{i=1}^k A \left(n - s_i - \sum_{j=1}^{p_i} A(n - a_{ij}) \right) \quad \text{and} \quad B(n) = \sum_{i=1}^k B \left(n - s_i - \sum_{j=1}^{p_i} B(n - a_{ij}) \right).$$

Define a sequence $C(n)$ as follows: $C(1), \dots, C(2r) = A(1), B(1), A(2), B(2), \dots, A(r), B(r)$ and for $n > r$, $C(2n - 1) = 2A(n)$ and $C(2n) = 2B(n)$. Then $C(n)$ is a solution of the “doubled” recursion

$$\langle 2s_1; 2a_{11}, 2a_{12}, \dots, 2a_{1p_1} : 2s_2; 2a_{21}, 2a_{22}, \dots, 2a_{2p_2} : \dots : 2s_k; 2a_{k1}, 2a_{k2}, \dots, 2a_{kp_k} \rangle$$

of the form (1.1) with initial conditions $C(1), C(2), \dots, C(2r)$.

Proof. We show that for any $n > 2r$, $C(n) = \sum_{i=1}^k C \left(n - 2s_i - \sum_{j=1}^{p_i} C(n - 2a_{ij}) \right)$. We proceed by induction for the case n odd. The case n even is entirely analogous so we omit the details.

It is straightforward to verify that the result holds for $n = 2r + 1$. Assume that it holds for all odd n up to $n = 2m - 3 > 2r$. We show it holds for $n = 2m - 1$, where $m > r$.

Applying the definition of $C(n)$ and simple algebra yields

$$\begin{aligned} & \sum_{i=1}^k C \left(2m - 1 - 2s_i - \sum_{j=1}^{p_i} C(2m - 1 - 2a_{ij}) \right) = \sum_{i=1}^k C \left(2m - 1 - 2s_i - \sum_{j=1}^{p_i} C(2(m - a_{ij}) - 1) \right) \\ &= \sum_{i=1}^k C \left(2m - 1 - 2s_i - \sum_{j=1}^{p_i} 2A(m - a_{ij}) \right) = \sum_{i=1}^k C \left(2(m - s_i - \sum_{j=1}^{p_i} A(m - a_{ij})) - 1 \right) \\ &= \sum_{i=1}^k 2A \left(m - s_i - \sum_{j=1}^{p_i} A(m - a_{ij}) \right) = 2 \sum_{i=1}^k A \left(m - s_i - \sum_{j=1}^{p_i} A(m - a_{ij}) \right) = 2A(m). \end{aligned}$$

Note that we make use of the fact that $m - a_{ij} < m$ on several occasions in the above induction argument.

⁴Compare this result with Theorem 4.2 in Section 4, where the resulting recursion has different order from the original one.

□

It is evident that an analogous result can be proved in a similar way for three or more solutions and sets of initial conditions.

Using this result we construct a solution $C(n)$ to a 2-ary order 1 recursion with the property that $\frac{C(n)}{n}$ does not have a limit; note that this technique generalizes to provide counterexamples of arbitrary order and arity. Let $A(n)$ satisfy the recursion $\langle 1; 1 : 3; 3 \rangle [0, 0, 0, 0]$, which generates a sequence of all zeroes, so $\frac{A(n)}{n}$ converges to 0. Let $B(n)$ satisfy the recursion $\langle 1; 1 : 3; 3 \rangle [1, 1, 1, 2]$, that is, the same recursion but with different initial conditions. The solution $B(n)$ is the slow sequence in which all numbers that are not powers of two appear twice, and all numbers that are powers of two appear three times, so $\frac{B(n)}{n}$ converges to $\frac{1}{2}$ (for a proof of these facts about $B(n)$ see [4]). Applying Theorem 2.2 with the solutions $A(n)$ and $B(n)$ to the recursion $\langle 1; 1 : 3; 3 \rangle$ results in the solution $C(n)$ to the recursion $\langle 2; 2 : 6; 6 \rangle$. The even subsequence gives $\limsup \frac{C(n)}{n} = \frac{1}{2}$ and the odd subsequence yields $\liminf \frac{C(n)}{n} = 0$. Therefore the sequence $\frac{C(n)}{n}$ has no limit.

3. ORDER 1 CONOLLY-LIKE RECURSIONS

Recall from Table 1 that the only Conolly-like sequences that might solve a 2-ary, order 1 recursion of the form (1.3) are (0, 1)-Conolly and (2, 0)-Conolly. We begin by examining which order 1 recursions are (0, 1)-Conolly.

Aside from the original Conolly recursion $\langle 0; 1 : 1; 2 \rangle [1, 2]$ our detailed computer search identified only one candidate, namely, $\langle 0; 2 : 3; 5 \rangle [1, 2, 2, 3, 4]$. We now prove that the Conolly sequence satisfies this recursion. We conjecture that this is the only other order 1 (0, 1)-Conolly recursion.

Theorem 3.1. *Both of the order 1 recurrences*

$$C(n) = C(n - C(n - 1)) + C(n - 1 - C(n - 2))$$

with $C(1) = 1, C(2) = 2$ and

$$H(n) = H(n - H(n - 2)) + H(n - 3 - H(n - 5))$$

with $H(1) = 1, H(2) = H(3) = 2, H(4) = 3, H(5) = 4$ have the Conolly sequence as their solution.

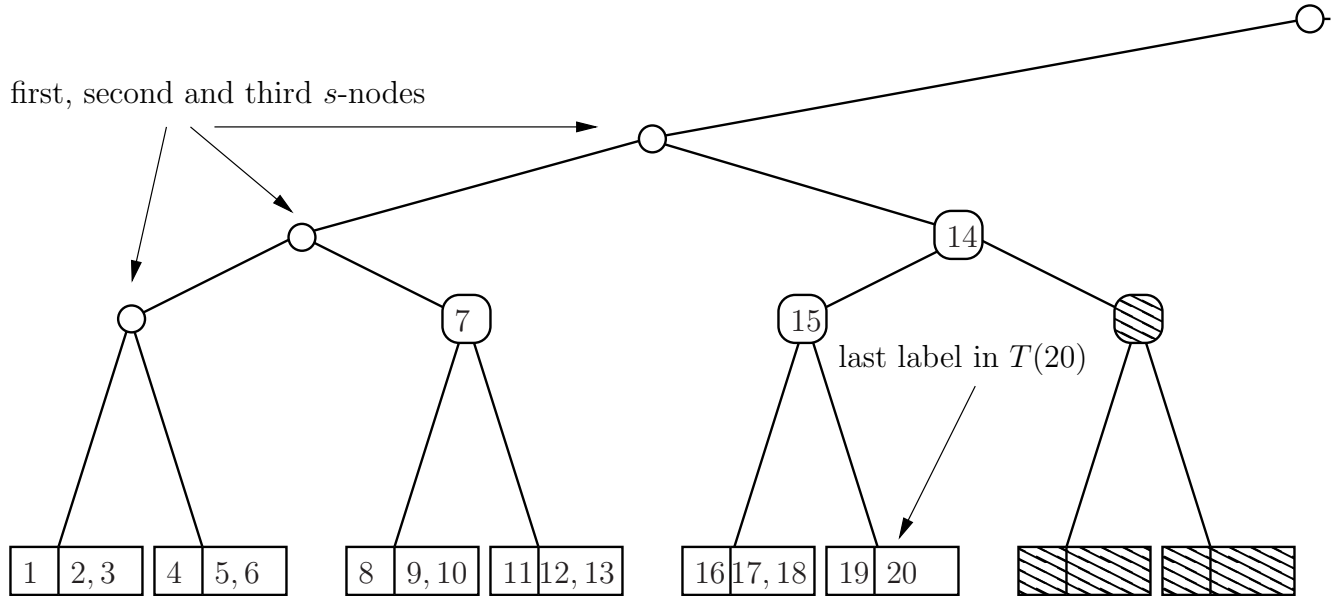
As we pointed out in the Introduction, it is proved combinatorially in [11] that the ruler function r_m is the frequency sequence of the Conolly sequence by exhibiting an explicit bijection between the Conolly sequence and the number of leaves of a labeled infinite binary tree. We elaborate on this methodology to show that the ruler function is also the frequency sequence of $H(n)$.

We begin by defining the following infinite binary tree T . We start with a sequence of nodes called s -nodes so that for each $m > 0$, the m^{th} s -node is the root of a complete binary tree of height $m + 1$, with the $(m - 1)^{\text{st}}$ s -node as its left child. Call the nodes on the bottom level *leaves*. Each leaf contains two cells,

cell 1	cell 2
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. The nodes which are neither leaves nor s -nodes are called *regular* nodes.

Define $T(n)$ to be T with the n labels $1, 2, \dots, n$ inserted in T in the following way: label the nodes of T in pre-order, beginning with the leftmost leaf. Regular nodes receive one label, s -nodes receive no labels, and leaves receive three consecutive labels, the left cell receiving

FIGURE 1. The infinite binary tree $T(20)$.

the first label and the right cell receiving the following two labels. For example, if a left leaf child has labels $\boxed{a+1} \boxed{a+2, a+3}$, then its parent (at the penultimate level) must be labeled with a while its right sibling has labels $\boxed{a+4} \boxed{a+5, a+6}$ so long as there are sufficient labels (it may occur that there are insufficient labels to fully label the final leaf). See Figure 1. A cell is considered *non-empty* if it contains at least one label.

TABLE 2. The sequence $L(n)$ for $n = 1, 2, \dots, 20$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$L(n)$	1	2	2	3	4	4	4	5	6	6	7	8	8	8	8	9	10	10	11	12

Let $L(n)$ be the number of non-empty cells in $T(n)$. See Table 2 for the first twenty terms of the sequence $L(n)$, which can readily be computed from $T(20)$ in Figure 1.⁵ Our strategy is to show that for all n , $L(n) = H(n)$ and that $L(n)$ is the Conolly sequence. Note that $L(n)$ has the same 5 initial values as $H(n)$, namely, 1, 2, 2, 3, 4. For $n > 5$ we show that $L(n)$ satisfies the same recursion as $H(n)$, namely, $L(n) = L(n - L(n - 2)) + L(n - 3 - L(n - 5))$.

To do so we extend the general technique described in [10]. We define a pruning operation that transforms the tree $T(n)$ with n labels and $L(n)$ cells into the tree $T(n - L(n - 2))$ with $n - L(n - 2)$ labels and $L(n - L(n - 2))$ cells. Before doing so we digress to establish a required technical result relating $L(n)$, the number of cells in $T(n)$, and $L(n - 2)$. Our strategy is to use the fact that the tree $T(n - 2)$ is obtained from $T(n)$ by removing the last two labels, $n - 1$ and n .

⁵It is evident that for $m \leq n$ $T(m)$ is a sub-tree of $T(n)$.

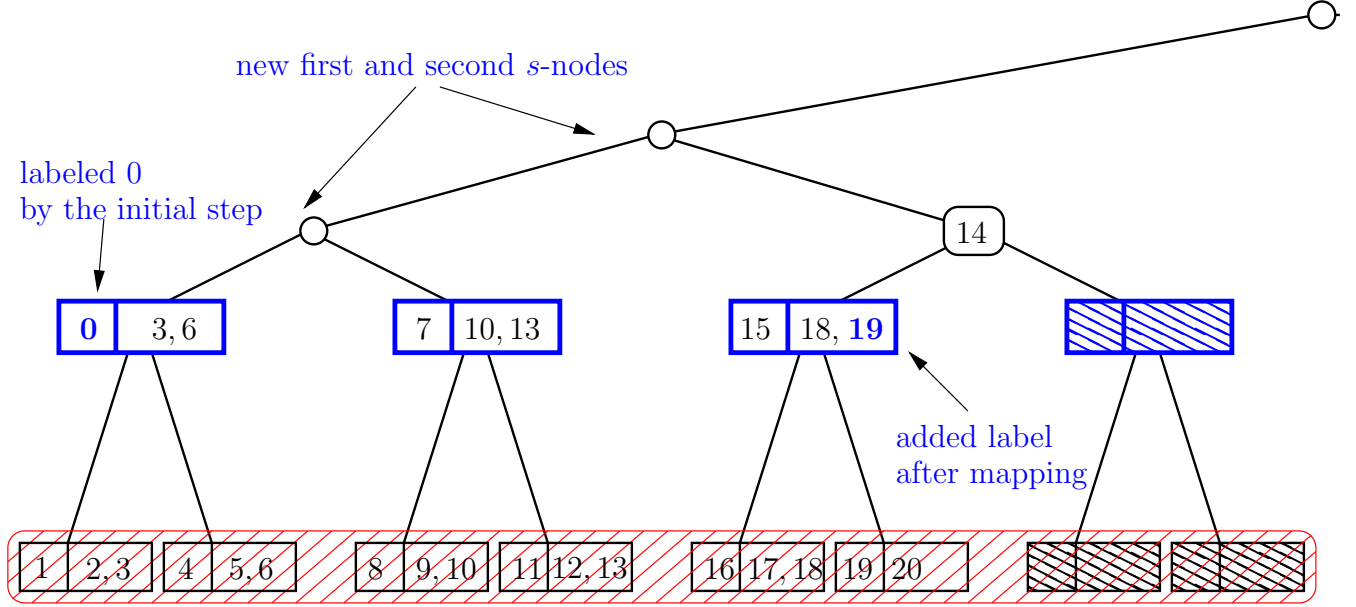


FIGURE 2. The pruning of $T(20)$ before the final relabeling. After relabeling this tree will be $T(10)$.

Lemma 3.1. *For all $n \geq 3$,*

$$L(n-2) = \begin{cases} L(n) & \text{if } n \text{ is on a regular node,} \\ L(n) - 1 & \text{if } n \text{ is the first label on a leaf,} \\ L(n) - 2 & \text{if } n \text{ is the second label on a leaf,} \\ L(n) - 1 & \text{if } n \text{ is the third label on a leaf.} \end{cases}$$

Proof. If n is on a regular node, then $n - 1$ is either on a regular node or is the third label on a leaf (equivalently, the second label in the second cell of the leaf). Therefore, removing n and $n - 1$ cannot empty any cells so $L(n - 2) = L(n)$. If n is the first label on a leaf, then again $n - 1$ is either on a regular node or is the third label on a leaf. Thus removing n and $n - 1$ empties the cell containing n but no others. Hence $L(n - 2) = L(n) - 1$. If n is the second label on a leaf, so is the first label in the second cell of the leaf, then $n - 1$ is the first label on that leaf and the only label in the first cell. Removing n and $n - 1$ empties both the cells in which they are contained, so $L(n - 2) = L(n) - 2$. Finally, if n is the third label on a leaf, then $n - 1$ and n are the two labels on that leaf's second cell, so removing these labels results in emptying only that cell. In this final case $L(n - 2) = L(n) - 1$. \square

The use of the above lemma is as follows. We want to turn $T(n)$ into $T(n - L(n - 2))$. The naive way to do this would be to simply remove the last $L(n - 2)$ labels in preorder, but we cannot prove anything useful about the resulting tree with this pruning method. Rather, we exploit the fact that since $L(n)$ is the number of nonempty cells of $T(n)$, it follows that $T(n - L(n))$ is just $T(n)$ with one label deleted from every non-empty cell. We then use the above lemma to adjust $T(n - L(n))$ into $T(n - L(n - 2))$, which together with other small changes preserves the original cell structure of T . In what follows we explain the precise details of this process.

We break down our pruning operation into several discrete steps that, when combined, transform $T(n)$ to get to $T(n - L(n - 2))$. See Figure 2 for an illustration with $n = 20$. Begin by labeling the first s -node with the label 0, thereby making it a regular node at the penultimate level (the *initial step*). Next, delete $L(n)$ labels from $T(n)$, one label from every non-empty cell (the *deletion step*). In each of the penultimate level nodes of the resulting tree, create two cells, inserting the existing label from that node in the first of these cells (the *cell creation step*). Empty all the leaves by moving any remaining leaf labels (these labels must be in the second cell) to the second cell in their respective parent at the penultimate level, and then delete all the (empty) leaves (the *lifting step*). Finally, following from Lemma 3.1, in some cases we need to add or subtract one label to be sure we end up with exactly $n - L(n - 2)$ labels (recall that we have already added one label in the initial step). If n was on a regular node in $T(n)$, we remove the final label. If n was the first or third label on a leaf in $T(n)$ we make no further changes. If n was the second label on a leaf in $T(n)$, we add one label in the next available position in preorder (the *correction step*).

We now check that the tree resulting from the pruning operation is $T(n - L(n - 2))$. It is clear that the tree has the correct structure so we need only confirm that it contains the correct number of labels. Before the pruning operation $T(n)$ contained n labels. After the initial step there are $n + 1$ labels. Following the deletion step there are $n + 1 - L(n)$ labels. By Lemma 3.1 the labels, if any, introduced or deleted in the correction step result in a total of $n - L(n - 2)$ labels. Notice that we could renumber the labels of this tree from 1 to $n - L(n - 2)$ but since only the number of labels matters we omit this step.

Theorem 3.2. *If $n \geq 6$ then the number of nonempty cells in the left leaves of $T(n)$ is $L(n - L(n - 2))$ and the number of nonempty cells in the right leaves of $T(n)$ is $L(n - 3 - L(n - 5))$. Hence $L(n) = L(n - L(n - 2)) + L(n - 3 - L(n - 5))$.*

Proof. We first show that the number of non-empty cells in the left leaves of $T(n)$ is $L(n - L(n - 2))$. Consider an arbitrary nonempty penultimate level node X of $T(n)$ with the label a . We show that after the pruning process is complete, X will have the same number of nonempty cells as its left child had before the pruning process. We consider 5 cases.

Case 1: $n = a$. In this case after the cell creation step in the pruning process X will be labeled $\boxed{a \quad}$. Since X has empty children, after the lifting step, the new labeling on X will remain unchanged. Since X is the last non-empty node in preorder for the correction step, and since initially n was on the regular node X , the correction step deletes the label a , leaving X empty. Thus, after the pruning process X has no non-empty cells, as required.

Case 2: $n = a + 1$. Here the left child of X is labeled $\boxed{a + 1 \quad}$; the right child is empty. After the lifting step, X will be labeled $\boxed{a \quad}$, and the correction step does nothing because n was the first label on a leaf. Thus, after the pruning process X has one nonempty cell, corresponding to the one non-empty cell that its left child had before the pruning process.

Case 3: $n = a + 2$. In this case the left child of X is labeled $\boxed{a + 1 \quad a + 2}$ and the right child is empty. So, after the lifting step, X will be labeled $\boxed{a \quad}$. Since X is now last in preorder, and n was the second entry in a leaf, the correction step adds one label to X , causing it to have an entry in its second cell after the pruning process. Thus X has two non-empty cells, corresponding to the two non-empty cells that its left child had before the pruning process.

Case 4: $n = a + 3, a + 4$ or $a + 5$. In all of these cases, the left child of X is full, with labels $\boxed{a + 1 \quad a + 2, a + 3}$ while the right child of X has at most two labels so at most one

label per cell. After the lifting step, X is labeled $\boxed{a \mid a + 3}$. Since the correction step will either do nothing or add a second label to the already nonempty second cell of X , X ends the pruning process with two nonempty cells, as required.

Case 5: $n \geq a + 6$. In this final case, the left and right children of X are fully labeled as $\boxed{a + 1 \mid a + 2, a + 3}$ and $\boxed{a + 4 \mid a + 5, a + 6}$, respectively. Thus, after the lifting step, X is labeled $\boxed{a \mid a + 3, a + 6}$. The correction step may remove at most one label from X , namely, the label $a + 6$. But doing so does not empty the second cell of X , so X ends the pruning process with two nonempty cells.

We next show that the number of nonempty cells in the right leaves of $T(n)$ is $L(n - 3 - L(n - 5))$. First note that by substituting $n - 3$ for n in the above discussion we conclude that $L(n - 3 - L(n - 5))$ counts the number of nonempty cells in the left leaves of $T(n - 3)$. Since leaves of T can hold up to three labels, if we have a sibling pair consisting of a left and right leaf, then a cell in the left leaf will be nonempty in $T(n - 3)$ if and only if the corresponding cell in the right leaf is nonempty in $T(n)$. Thus, the number of nonempty cells in the left leaves of $T(n - 3)$ is the same as the number of nonempty cells in the right leaves of in $T(n)$. This completes the proof. \square

We now show that $L(n)$ is the Conolly sequence, from which it follows that the frequency sequence for $L(n)$ is the ruler function.

Theorem 3.3. *The solution sequence $L(n)$ of the recursion*

$$H(n) = H(n - H(n - 2)) + H(n - 3 - H(n - 5))$$

with initial conditions $H(1) = 1, H(2) = H(3) = 2, H(4) = 3, H(5) = 4$ is the Conolly sequence.

Proof. We prove that $L(n)$ is the Conolly sequence by showing that both $L(n)$ and the Conolly sequence have the same first difference sequences. Define the finite binary string D_n recursively by the rules $D_0 = 1$ and $D_{n+1} = 0D_nD_n$ whenever $n \geq 0$. Then the infinite sequence of successive first differences $C(n + 1) - C(n)$ of the Conolly sequence $C(n)$ is given by

$$\mathcal{D} = D_0D_0D_1D_2D_3 \cdots ,$$

with the convention that $C(0) = 0$ so the difference sequence starts with $C(1) - C(0) = 1$ (see [11]).

Let $d(n) = L(n) - L(n - 1)$. Then $d(n)$ is the increase in the number of non-empty cells when applying an n th label to $T(n - 1)$, where we take $T(0)$ to be the unlabeled tree T , that is, $T(0) = T$ and $L(0) = 0$. Define the binary string F_n by the rules $F_1 = 110110$, $F_2 = 0F_1$ and $F_{n+1} = 0F_nF_n$ whenever $n \geq 2$. We show by induction that the infinite binary sequence

$$\mathcal{F} = F_1F_2F_3 \cdots$$

gives the sequence $d(1)d(2)d(3) \dots$

To do this, we introduce a new symbol. For all $m > 0$, let T_m be the subtree of T consisting of the right child of the m^{th} s -node and all the descendants of that right child.

We first show that $F_1F_2 = d(1)d(2) \cdots d(13)$. The string F_1 gives the subsequence of $d(n)$ corresponding to any pair of leaf siblings in $T(n)$ (and hence $d(1)d(2) \cdots d(6) = F_1$). The right child of the second s -node of T is the root of T_2 , a 3-node complete binary tree and clearly $d(7)d(8) \cdots d(13) = 0F_1 = F_2$, so F_2 is the subsequence of $d(n)$ corresponding to T_2 .

We now show inductively that for $m \geq 2$, F_m is the subsequence of $d(n)$ corresponding to T_m . We already have the base case, since F_2 is the subsequence of $d(n)$ corresponding to T_2 . Assume that F_{m-1} is the subsequence of $d(n)$ corresponding to T_{m-1} . The key observation is that for $m \geq 3$, T_m consists of a single regular node from which descend two copies of T_{m-1} . The preorder traversal of T_m begins with the regular node at its root (corresponding to a 0 in the difference sequence) followed by the left copy of T_{m-1} (corresponding to one repetition of F_{m-1} in the difference sequence by the induction hypothesis) followed by the right copy of T_{m-1} (hence another repetition of F_{m-1} in the difference sequence). Therefore the part of the difference sequence corresponding to T_m is $F_m = 0F_{m-1}F_{m-1}$. Since the sequence of subtrees T_m are labeled successively, and the s -nodes that join them in pairs are empty, $d(1)d(2)\cdots = \mathcal{F}$.

Finally, we show that $\mathcal{F} = \mathcal{D}$, from which we conclude that $L(n)$ matches the Conolly sequence. In what follows, by the notation 0^{-1} we mean the inverse of 0 in the free group on the symbols 0, 1 which make up these binary strings. Similarly, 1^2 means the string 11, and so on. Observe that $F_1 = D_0D_0D_10$. For $n > 1$, we show by induction that $F_n = 0^{-1}D_n0$, from which we get that $\mathcal{F} = \mathcal{D}$. For $n = 2$ this can be verified directly: $F_2 = 01^201^20 = 0^{-1}0^21^201^20 = 0^{-1}0D_1D_10 = 0^{-1}D_20$.

For $n \geq 2$ assume $F_n = 0^{-1}D_n0$. Then $F_{n+1} = 0F_nF_n = 00^{-1}D_n00^{-1}D_n0 = D_nD_n0 = 0^{-1}0D_nD_n0 = 0^{-1}D_{n+1}0$ as required. This completes the induction. \square

Recall from Table 1 that there are only two possibilities for the values of the (α, β) pairs for Conolly-like solutions of 2-ary, order 1 meta-Fibonacci recursions, namely, $(\alpha, \beta) = (0, 1)$ or $(2, 0)$. We conclude our discussion of such recursions by identifying all the $(2, 0)$ -Conolly 2-ary, order 1 meta-Fibonacci recursions. Note that the $(2, 0)$ -Conolly solution to these recursions is by definition the ceiling sequence $\lceil \frac{n}{2} \rceil$ (see [4] for a special case). Our characterization of these recursions follows immediately from a more general result (see Theorem 5.2 below).

Corollary 3.1. *The function $\lceil \frac{n}{2} \rceil$ is the unique solution that satisfies the 2-ary, order 1 recursion $R = \langle s; a : t; b \rangle$, given sufficiently many initial conditions $R(z) = \lceil \frac{z}{2} \rceil$, if and only if a and b are both odd, and $2(s + t) = a + b$.*

For any given set of parameters $\{s, a, t, b\}$ with a and b both odd and $2(s + t) = a + b$, it is easy to show that

$$m = \max \left\{ a, b, s + \frac{a+1}{2}, t + \frac{b+1}{2} \right\}$$

initial conditions are sufficient. This upper bound is not tight, however. For example, if the parameters are 1, 3, 3, 5 then the initial conditions 1, 1, 2, 2, 3 are sufficient, but $m = 6$ in this case.

4. CONOLLY-LIKE RECURSIONS OF HIGHER ORDER

From Sections 2 and 3 we have necessary and sufficient conditions for the existence of a 2-ary, order 1 (α, β) -Conolly recursion. In this section we extend this result to recursions of all orders using a proof strategy similar to that of Theorem 3.2. For this purpose we invent a new labeling of the same tree T defined in Section 3.

First, we state our key result:

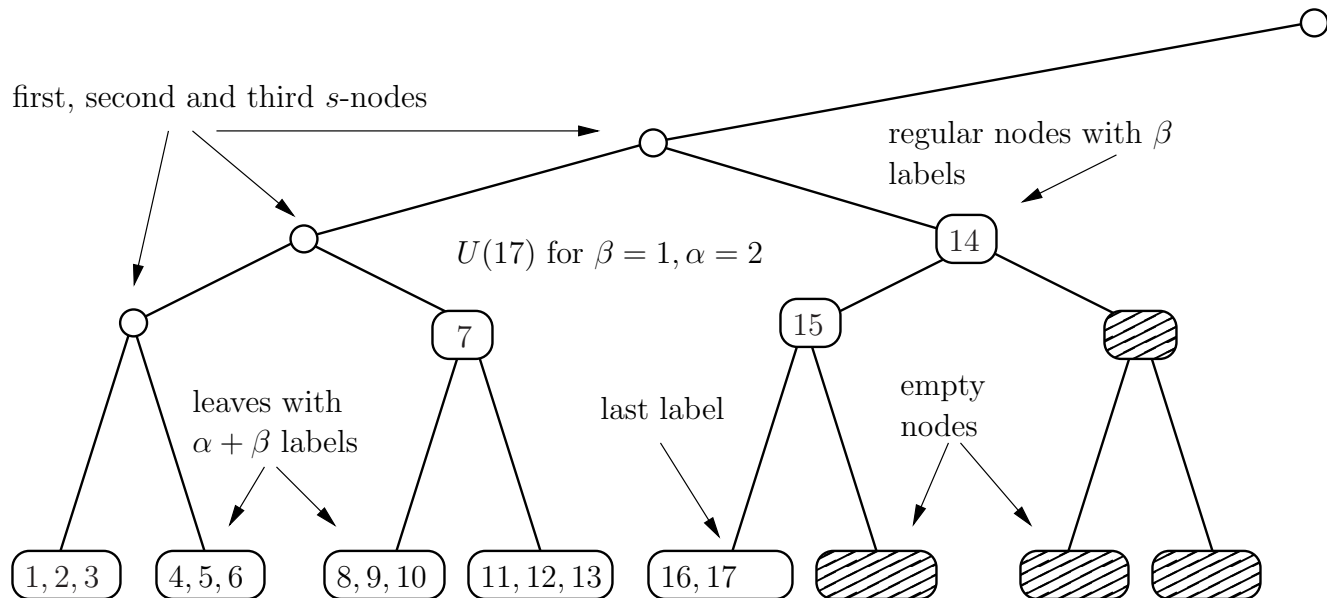


FIGURE 3. The labeled tree $U(17)$ with $\alpha = 2$ and $\beta = 1$. We see that $M(17) = 5$.

Theorem 4.1. *There exists a 2-ary (α, β) -Conolly meta-Fibonacci recursion if and only if $\beta \geq 0, \alpha + \beta > 0$, and α is even.*

Observe that the “only if” part of Theorem 4.1 follows directly from Corollary 2.1. To prove sufficiency we claim that the recursion

$$(4.1) \quad \langle 0; 1, 3, 5, \dots, 2p - 1 : \gamma; \gamma + 1, \gamma + 3, \gamma + 5, \dots, \gamma + 2p - 1 \rangle$$

has an (α, β) -Conolly solution, where $\gamma = \alpha + \beta$, and where it follows from Corollary 2.1 that p must be $\alpha/2 + \beta$.

To establish our claim we first describe the infinite binary tree U that provides the required combinatorial interpretation for the solution to the above recursion, as well as the pruning process that we apply to U . The tree U has the same general structure as the tree T used in Theorem 3.2, that is, U has s -nodes, regular nodes, and leaves in the same locations as T . As with T , the s -nodes of U are always empty. However, there are some important differences from T in the way that we label the remaining nodes of U . The regular nodes of U contain up to β labels (rather than 1 label, as in T). The leaves of U do not have multiple cells; each leaf of U can contain a total of $\alpha + \beta$ labels (rather than a maximum of 3 labels in T).

Define $U(n)$ to be U with the n labels 1 through n assigned in preorder. As was the case with $T(n)$, we refer to nodes of $U(n)$ that have no labels as “empty,” and nodes in $U(n)$ containing their maximum number of labels as “full.” Only the last nonempty node in $U(n)$, the one containing the label n , can be partly filled. Analogous to the function $L(n)$ in Theorem 3.2, define $M(n)$ to be the number of nonempty leaf nodes on $U(n)$. Unlike L , since the leaves of $U(n)$ do not have multiple cells, M counts leaves, not cells. See $U(17)$ in Figure 3 where $\alpha = 2$ and $\beta = 1$.

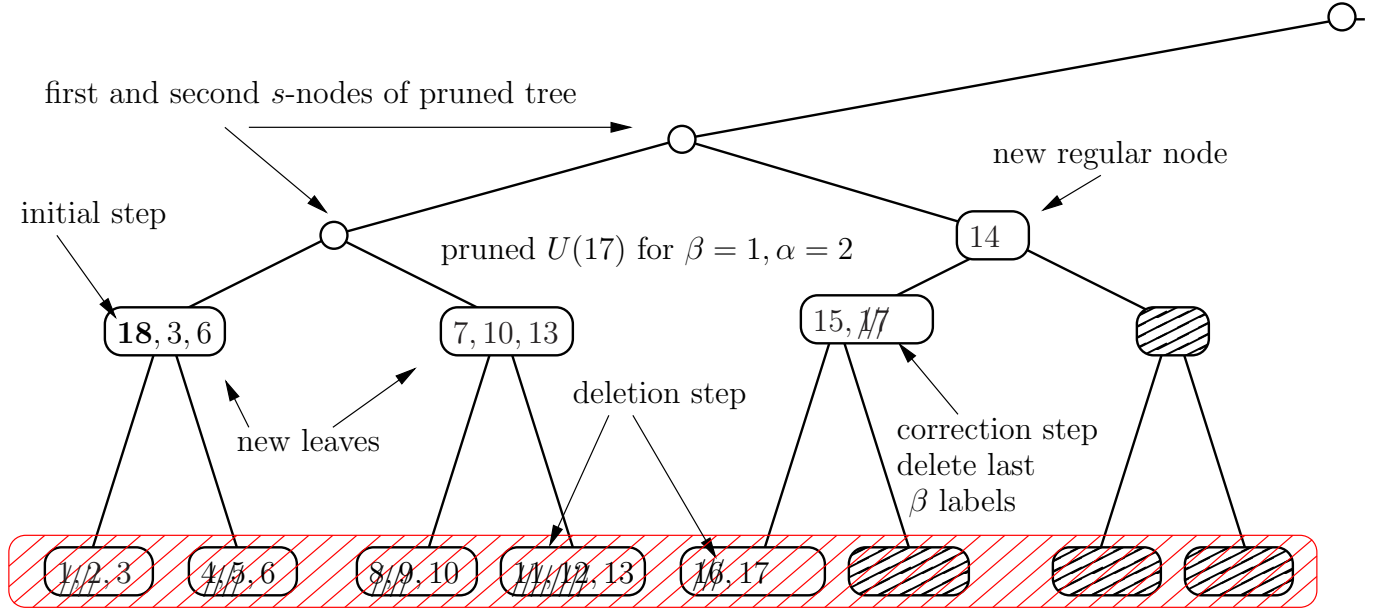


FIGURE 4. The pruning of $U(17)$ with $\alpha = 2$ and $\beta = 1$ before the final relabeling. After relabeling this tree will be $U(8)$.

We define a pruning operation on the tree $U(n)$ which transforms $U(n)$ into the tree $U(n - \sum_{j=0}^{j=p} M(n - 2j + 1))$. Because of the much greater generality of the result that we seek to prove here this pruning operation is considerably more complex than the one we described above for $T(n)$. We proceed with the explanation in stages, beginning with $\alpha \geq 0$.

The *initial step* is as follows: take $U(n)$ and add β labels to the first s -node, thereby making this node identical to the other full regular nodes at the penultimate level (now the tree has $n + \beta$ total labels). Next, the *deletion step*: for each leaf Y of $U(n)$, remove one label from Y for each of the trees $U(n - 1), U(n - 3), \dots, U(n - 2p + 1)$ in which Y is not an empty node. For example, if the first label on Y is a , and $n \geq a + 2p - 1$, then Y is necessarily full in $U(n)$ and nonempty in all p of the trees $U(n - 1), U(n - 3), \dots, U(n - 2p + 1)$. Thus, in this case the deletion step removes p labels from Y . More generally, if Y is any full leaf, there are certainly at least p labels available to remove since $\alpha \geq 0$ and $p = \alpha/2 + \beta \leq \alpha + \beta$; if Y is not full then by Lemma 4.1 below we will not delete all the labels of Y . Our tree now has $n + \beta - \sum_{j=1}^{j=p} M(n - 2j + 1)$ labels.

Notice that after the above deletion step, some of the leaves may still have labels in them (at most $\alpha/2$ labels). We deal with these leftover labels in the *lifting step*: we take any such labels and lift them into the parent of their leaf node in U . Complete the lifting step by deleting all of the (now empty) leaves.

We complete the pruning process with the *correction step*: delete the last β labels (by preorder) in the tree. It is clear from the construction that our final pruned tree has $n - \sum_{j=1}^{j=p} M(n - 2j + 1)$ labels and is $U(n - \sum_{j=1}^{j=p} M(n - 2j + 1))$, up to renumbering of labels. See Figure 4.

We now describe how to prune $U(n)$ when $\alpha < 0$. In this case the leaves contain fewer than p labels. We follow the same pruning process as above, except when it would require

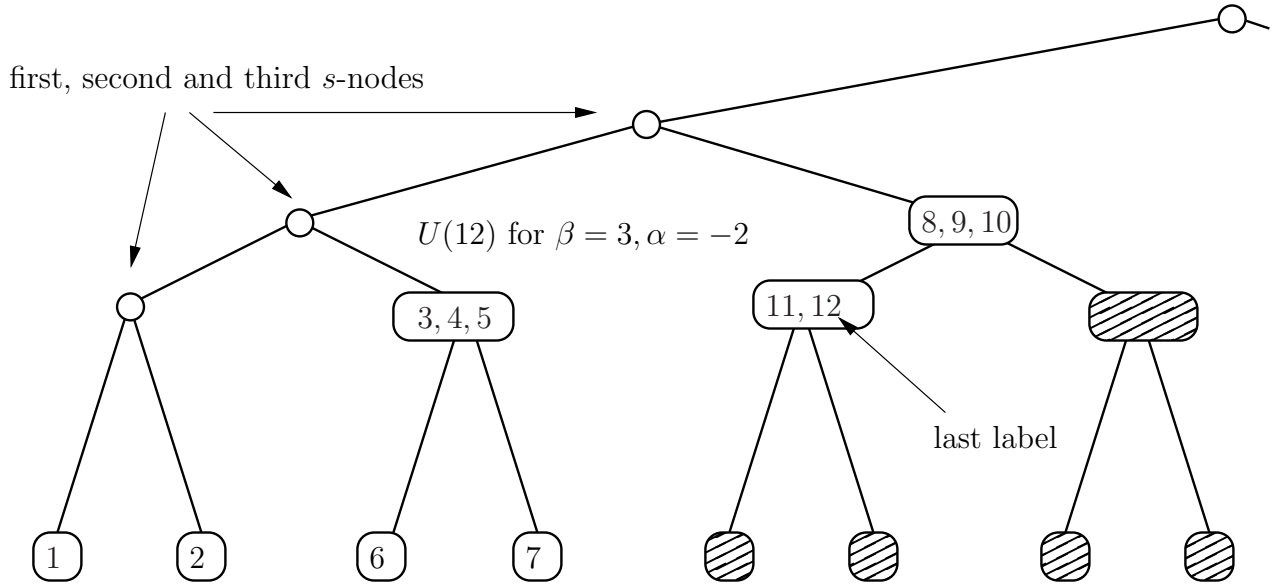


FIGURE 5. The labeled tree $U(12)$ with $\alpha = -2$ and $\beta = 3$.

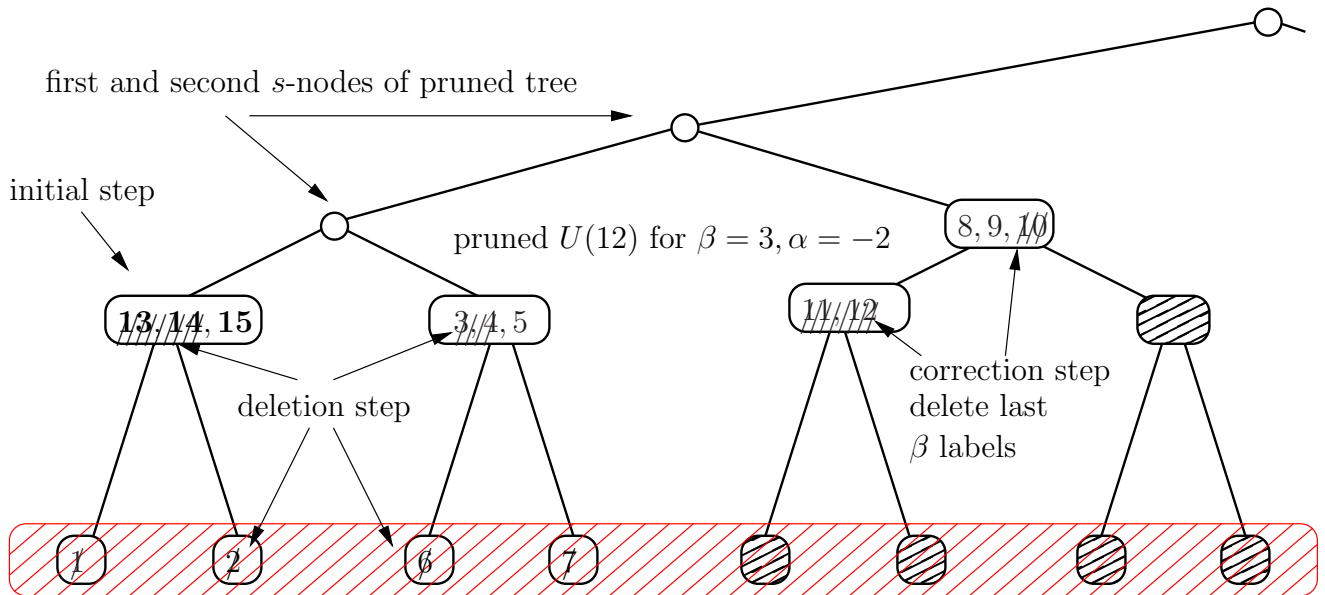


FIGURE 6. The pruning of $U(12)$ with $\alpha = -2$ and $\beta = 3$ before the final relabeling. After relabeling this tree will be $U(4)$.

that we delete more labels from a leaf node than that node contains. If the leaf Y has a deficit δ in the number of available labels to delete, then we delete all the labels from Y and δ labels from the parent of Y . Note that since $\delta \leq -\alpha/2$, we delete at most $-\alpha$ labels from the parent of each leaf. Since $\alpha + \beta > 0$, we always have enough labels in the penultimate level nodes for this to be possible. See Figures 5 and 6.

Analogous to Lemma 3.1, we count the number of labels that are deleted from any given leaf in the following lemma.

Lemma 4.1. *Let Y be a leaf node of $U(n)$. Suppose there are d labels on or after Y in preorder (that is, labels either on Y or on nodes which are further along in preorder than Y). Then during the deletion step of the pruning process, $\min\{\lfloor d/2\rfloor, p\}$ labels are deleted from Y .*

Proof. Clearly, if $\lfloor d/2\rfloor > p$, then p labels are deleted from Y since Y is nonempty on $U(n - 2i + 1)$ for $i \leq p$. Otherwise, since $2i - 1 < d$ if and only if $i \leq \lfloor d/2\rfloor$, we will have at least one label on or after Y for all the trees $U(n - 2i + 1)$ for $i \leq \lfloor d/2\rfloor$. Thus $\lfloor d/2\rfloor$ labels will be deleted from Y . \square

We are now prepared to prove Theorem 4.1.

Proof. As discussed above, it suffices to show that (4.1) has an (α, β) -Conolly solution.

First, we show that $M(n)$ is an (α, β) -Conolly sequence. If $\alpha = 0, \beta = 1$, then U is identical to the tree studied in [11], where it is shown that $M(n)$ is the Conolly sequence. For $\alpha = 0$ and $\beta > 1$, then every node holds β times as many labels as in the tree in [11], which multiplies the number of times each integer appears in the sequence $M(n)$ by β ; the result is a $(0, \beta)$ -Conolly sequence. Increasing or decreasing α changes only the number of labels per leaf. This means that every integer will appear an additional α times (or $-\alpha$ fewer times), giving an (α, β) -Conolly sequence.

Next, we show that

$$\langle 0; 1, 3, 5, \dots, 2p - 1 : \gamma; \gamma + 1, \gamma + 3, \gamma + 5, \dots, \gamma + 2p - 1 \rangle,$$

with initial conditions $M(1), \dots, M(4\alpha + 5\beta)$, is $M(n)$. This will establish the theorem. Note that $4\alpha + 5\beta$ is the last label on the 4^{th} leaf of U .

First we prove that for all $n > 4\alpha + 5\beta$, $M(n - \sum_{j=1}^p M(n - 2j + 1))$ counts the number of left leaves of $U(n)$. Recall that the above definition of pruning transforms the tree $U(n)$ into the tree $U(n - \sum_{j=1}^{j=p} M(n - 2j + 1))$ (up to renumbering of the labels). Thus, $M(n - \sum_{j=1}^p M(n - 2j + 1))$ counts the number of nonempty leaves of $U(n - \sum_{j=1}^{j=p} M(n - 2j + 1))$. We exhibit a bijection between the nonempty leaves of $U(n - \sum_{j=1}^{j=p} M(n - 2j + 1))$ and the nonempty *left* leaves of $U(n)$; equivalently, we show that in the pruning process that transforms $U(n)$ into $U(n - \sum_{j=1}^{j=p} M(n - 2j + 1))$, a given penultimate level node X of $U(n)$ will be nonempty in $U(n - \sum_{j=1}^{j=p} M(n - 2j + 1))$ if and only if X has a nonempty left leaf child in $U(n)$. To do so we consider 4 cases corresponding to the position of the last label n on $U(n)$. For any penultimate node X we write Y (respectively, Z) for the left (respectively, right) leaf child of X .

Case 1: The label n is the d^{th} label on the left leaf child Y of X . In this case, we need to prove that X ends the pruning process with at least one label. By Lemma 4.1, the deletion step will remove at most $\lfloor d/2\rfloor < d$ labels from Y , so Y will end the deletion step with at least one label. Thus, in the lifting step at least one label shifts up from Y to X , causing X to have more than β labels after the lifting step. The correction step of the pruning process deletes the last β labels in preorder, and these labels will be on X . Since X now has more than β labels, it will end the pruning process with at least one label.

Case 2: The label n is the d^{th} label on the right leaf child Z of X . Again, we need to prove that X ends the pruning process with at least one label. Note that Y must be full since Z is

nonempty. As in Case 1, by Lemma 4.1 the deletion step removes at most $\lfloor d/2 \rfloor < d$ labels from Z , so we do not delete every label of Z . Furthermore, since there are $\alpha + \beta + d$ labels on or after Y and $d \leq \alpha + \beta$ labels on Z , there are at most $\lfloor (\alpha + \beta + d)/2 \rfloor \leq \alpha + \beta$ labels deleted from Y during the deletion step. Since there is no label deficit on Y , no labels need to be deleted from X . Therefore, before the lifting step, X will have β labels and Z will have at least one label. Thus, after the lifting step, X will have more than β labels. The correction step of the pruning process deletes the last β labels in preorder, and these labels will be on X . Since X now has more than β labels, it will end the pruning process with at least one label.

Case 3: The label n is on X . Since X has an empty left child, we want to show that X ends the pruning process without any labels. But n is the last label in preorder, and since X has no children it will not have any labels removed during the deletion step or added during the lifting step. Thus, the β labels that are removed from the tree in the correction step of the pruning process will come out of X first. But X has at most β labels. Therefore, after the pruning process, X will have no labels.

Case 4: The label n is after Z in preorder. We show that after the pruning process X retains at least one label. To do so we must consider 3 subcases.

Subcase 4.1: There are d labels on $U(n)$ after the last label of Z , where $1 \leq d < \beta$. Since a regular node contains up to β labels, it follows that these d labels must all be on the regular node, call it X' , that follows Z in preorder; note that X' has no non-empty children.

In this case, there are exactly $\alpha + \beta + d$ labels in preorder that are on or after Z and $2(\alpha + \beta) + d$ labels in preorder that are on or after Y . By Lemma 4.1, the number of labels deleted during the deletion step from Z (resp. Y) is $\lfloor (\alpha + \beta + d)/2 \rfloor = \alpha/2 + \lfloor (\beta + d)/2 \rfloor$ (resp. $\lfloor (2(\alpha + \beta) + d)/2 \rfloor = \alpha + \beta + \lfloor d/2 \rfloor$). Further, since X' has no children, X' will have no labels added or deleted during the deletion step and the lifting step. In the correction step, we remove the last β labels from X' and X in our transformed tree. We now count labels carefully: when the pruning process began $U(n)$ had $2\alpha + 3\beta + d$ labels on or after X (including the β labels on X).

From the calculations in the above paragraph, during the the deletion and correction steps we deleted $(\alpha/2 + \lfloor (\beta + d)/2 \rfloor) + (\alpha + \beta + \lfloor d/2 \rfloor) + \beta = \frac{3}{2}\alpha + 2\beta + \lfloor (\beta + d)/2 \rfloor + \lfloor d/2 \rfloor$, labels in total, leaving us with $\alpha/2 + \beta + d - \lfloor (\beta + d)/2 \rfloor - \lfloor d/2 \rfloor \geq \alpha/2 + \beta + d - \beta/2 - d/2 - d/2 = \alpha/2 + \beta/2 > 0$, where in the last step we use that $\alpha + \beta > 0$ by assumption. Therefore, after the pruning process X has at least one label.

Subcase 4.2: There are d labels on $U(n)$ after the last label of Z , where $d \geq \beta$, and the next regular node X' in preorder after Z is not a penultimate level node. In the deletion step we remove at most $p = \alpha/2 + \beta$ labels from Y and from Z (by Lemma 4.1). If $\alpha \geq 0$ then either at least $\alpha/2$ labels are left on each of the leaves Y and Z ; these labels are lifted up to X in the lifting step. If $\alpha < 0$ then at most $-2\alpha/2 = -\alpha < \beta$ labels are removed from X . Either way, immediately prior to the correction step that removes β labels from the tree, X is nonempty.

By assumption, at the start of the pruning process there are β labels on X' . Since X' is neither a leaf node nor a penultimate level node, X' still has β labels immediately before the correction step. Since X' is after X in preorder, the β labels on X' are in line to be removed in the correction step before any of the labels in X . But only β labels are removed in the correction step, hence none of these labels come from X . Thus at the end of the pruning process X has at least one label.

Subcase 4.3: there are d labels on $U(n)$ after the last label of Z , where $d \geq \beta$ and the next regular node X' in preorder after Z is a penultimate level node. As in Subcase 4.2, immediately before the correction step, X will be nonempty. Observe that since the next regular node after the penultimate level node X is another penultimate level node X' , the next regular node after X' in preorder cannot be a third penultimate level node. Therefore, the penultimate level node X' must meet the requirements of one of Case 1, 2, 4.1 or 4.2. By the arguments of those cases, the correction step will not remove every label from X' . Therefore, the correction step cannot remove any labels from X , so after the pruning process X is nonempty.

We conclude that for all $n > 4\alpha + 5\beta$, $M(n - \sum_{j=1}^p M(n - 2j + 1))$ counts the number of left leaves of $U(n)$. Since the leaves of $U(n)$, other than possibly the last, have exactly $\gamma = \alpha + \beta$ labels, it follows that we can replace n by $n - \gamma$ in the above argument to conclude that n is on or after a right leaf in $U(n)$ if and only if $n - \gamma$ is on or after the sibling left leaf. This immediately implies that $M(n - \gamma - \sum_{j=1}^p M(n - 2j + 1 - \gamma))$ counts the number of right leaves of $U(n)$. Thus, $M(n - \sum_{j=1}^p M(n - 2j + 1)) + M(n - \gamma - \sum_{j=1}^p M(n - 2j + 1 - \gamma))$ counts the total number of leaves of $U(n)$, which equals $M(n)$. \square

This section concludes with two results that relate the solutions to low-order recursions to those of higher order. Let b be an integer sequence. Define the m -interleaving of b to be the sequence where each term of b is repeated m times. For example, if $b = 1, 2, 2, 3, \dots$ then the 3-interleaving of b is $1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, \dots$. We have new notation for this notion: if $b = b_1, b_2, b_3, \dots$ then let b_1^m denote b_1 repeated m times. The m -interleaving of b is thus written as $b_1^m, b_2^m, b_3^m, \dots$. Theorem 4.2 says that for any order 1 recurrence B , there is an order m recursion such that the m -interleavings of any solution to B solves A . These m -interleavings are useful, since the m -interleaving of the (α, β) -Conolly sequence is the $(m\alpha, m\beta)$ -Conolly sequence.

Theorem 4.2. (*Order Multiplying Interleaving Theorem*) *Let $B = \langle s; a : t; b \rangle[\xi_1, \xi_2, \dots, \xi_c]$ be a recurrence relation with a well defined solution sequence. Then for any integer $m > 1$ the recurrence*

$$A = \langle ms; (ma)^m : mt; (mb)^m \rangle[\xi_1^m, \xi_2^m, \dots, \xi_c^m]$$

also has a unique solution which is the m -interleaving of B , where superscript m denotes multiplicity.

Proof. We prove by induction on n that $A(mn - j) = B(n)$ for $0 \leq j < m$. Observe that for B to have a unique solution it must have at least one initial value ξ_1 . Thus for $n = 1$ we have the base case, $A(m - j) = \xi_1 = B(1)$ and similarly for all ξ_i . Assume the theorem is true for values less than n . We show it holds for n :

$$\begin{aligned} & A(mn - j) \\ &= A(mn - ms - j - mA(mn - ma - j)) + A(mn - mt - j - mA(mn - mb - j)) \\ &= A(m(n - s - A(m(n - a) - j)) - j) + A(m(n - t - A(m(n - b) - j)) - j) \\ &= A(m(n - s - B(n - a)) - j) + A(m(n - t - B(n - b)) - j) \\ &= B(n - s - B(n - a)) + B(n - t - B(n - b)) \\ &= B(n) \end{aligned}$$

Note that for the induction to be valid we must have $s + B(n - a) > 0$ and $t + B(n - b) > 0$. This is guaranteed because we require that $B(n) > 0$ for all $n > 0$, and $s, t \geq 0$. \square

If B is a slowly growing sequence and A is the sequence resulting from an application of Theorem 4.2, then certain perturbations of the parameters of A leave the solution unchanged. By taking B to be a $(0, 1)$ -Conolly recursion, we can produce a variety of $(0, m)$ -Conolly recursions by applying Theorem 4.2 to B and then perturbing the resulting recursion with Theorem 4.3. We could do the same with $(2, 0)$ -Conolly recursions, but in fact Theorem 5.2 provides a much stronger result in this case.

Theorem 4.3. *Let $B = \langle s; a : t; b \rangle [\xi_1, \xi_2, \dots, \xi_c]$ be a slowly growing sequence. Let $\alpha_1, \dots, \alpha_m$ and β_1, \dots, β_m be integer constants that satisfy for all $1 \leq i \leq m$*

$$(4.2) \quad \begin{aligned} i - m &\leq \alpha_i < i \text{ and} \\ i - m &\leq \beta_i < i. \end{aligned}$$

If the sequence

$$(4.3) \quad C = \langle ms; ma - \alpha_1, \dots, ma - \alpha_m : mt; mb - \beta_1, \dots, mb - \beta_m \rangle [\xi_1^m, \xi_2^m, \dots, \xi_c^m]$$

is well defined, then it is an m -interleaving of B .

Proof. Let A be the m -interleaving of B . We prove that

$$(4.4) \quad A \left(n - ms - \sum_{i=1}^m A(n - ma + \alpha_i) \right) = A(n - ms - mA(n - ma)).$$

and note that a similar argument works for

$$A \left(n - mt - \sum_{i=1}^m A(n - mb + \beta_i) \right) = A(n - mt - mA(n - mb)).$$

This will prove that $A = \langle ms; ma - \alpha_1, \dots, ma - \alpha_m : mt; mb - \beta_1, \dots, mb - \beta_m \rangle [\xi_1^m, \xi_2^m, \dots, \xi_c^m]$, by Theorem 4.2. Note that insisting C be well defined is simply a requirement that c is large enough so that $n - ma + \alpha_i > 0$ whenever $n > mc$.

The intuition behind this proof is that for each term referenced by B , the corresponding terms referenced by C lie in some m -interval, that is, they belong to $[km + 1, (k + 1)m]$ for some positive integer k . This provides flexibility in the parameters of C .

Let $n > mc$ be given, and set j such that $1 \leq j \leq m$ and $j = n \bmod m$. Now, observe that as A is an m -interleaving, its values can only change at multiples of m , that is, if $A(n) \neq A(n + 1)$, then $n = mz$ for some integer z . This means that in order to show (4.4), it suffices to show that $n - ms - \sum_{i=1}^m A(n - ma + \alpha_i)$ lies in the same m -interval as $n - ms - mA(n - ma)$. Observe that since j, n and $n - ms - mA(n - ma)$ are all equal mod m , the left and right endpoints of the m -interval containing $n - ms - \sum_{i=1}^m A(n - ma + \alpha_i)$ must be $n - ms - mA(n - ma) - (j - 1)$ and $n - ms - mA(n - ma) - j + m$. Thus we infer the following inequalities:

$$(4.5) \quad -mA(n - ma) - (j - 1) \leq -\sum_{i=1}^m A(n - ma + \alpha_i) \leq -mA(n - ma) + (m - j)$$

To do so, we first observe that since $i - m \leq \alpha_i < i$, adding α_i can only either move back one m -interval, not change the m -interval, or move forward one m -interval. Since $B(n)$ is

slowly growing, this means that $A_i := A(n - ma + \alpha_i) - A(n - ma) = 0, 1$, or -1 . We will show that A_i is 1 at most $j - 1$ times and -1 at most $m - j$ times, thereby establishing (4.5).

In order to show this, first observe that if n is in the interval $[km + 1, km + m]$, then $A_i = 1$ only if $n + \alpha_i$ is in the following interval $[km + m + 1, km + 2m]$, and $A_i = -1$ only if $n + \alpha_i$ is in the preceding interval $[km - m + 1, km]$. By the definition of j , $n = km + j$ so $n + \alpha_i$ is in the interval $[km + m + 1, km + 2m]$ if and only if $j + \alpha_i > m$, and $n + \alpha_i$ is in the interval $[km - m + 1, km]$ if and only if $j + \alpha_i \leq 0$.

Thus, the number of i with $A_i = 1$ is at most the number of i with $j + \alpha_i > m$, and the number of i with $A_i = -1$ is at most the number of i with $j + \alpha_i \leq 0$. So it suffices to show that at most $j - 1$ of the α_i satisfy $j + \alpha_i > m$, and at most $m - j$ of the α_i satisfy $j + \alpha_i \leq 0$.

If $j + \alpha_i > m$, then $\alpha_i > m - j$, so since $\alpha_i < i$, this can only be true for at most the $j - 1$ indices $m - j + 2, m - j + 3, \dots, m$. Similarly, if $j + \alpha_i \leq 0$, then since $\alpha_i \geq i - m$, it follows that $i - m \leq \alpha_i \leq -j$ so $i \leq m - j$, which means $i = 1, 2, \dots, m - j$, for a total of only $m - j$ values. This gives us the desired bound on the number of indices i with $A_i = 1$ or -1 , completing the proof. \square

5. ENUMERATING CONOLLY-LIKE RECURSIONS

Up to this point we have focused on showing that specific recursions are (α, β) -Conolly. Here we turn to the following question: for a given (α, β) , what is the complete list of (α, β) -Conolly recursions of the form (1.3)?

Based on our experimental evidence described in Section 2, we believe that for each possible (α, β) pair with $\beta > 0$ there are finitely many (and at least 1) 2-ary (α, β) -Conolly recursions. Below we illustrate this hypothesis in the case $p = 2$ (see Conjecture 5.1), where we list what we believe are all the 2-ary recursions. Note that the (α, β) pairs we list in Conjecture 5.1 appear in Table 1; in the tables in Conjecture 1 the set notation denotes that parameters can be chosen from the Cartesian product of the sets, while the right hand column in the tables is simply the size of that product.

Conjecture 5.1. *For $\beta > 0$, the only 2-ary, order 2 (α, β) -Conolly recurrences are:*

For $(\alpha, \beta) = (-2, 3)$:

<i>Recurrence</i>	<i>number</i>
$\langle 0; 1, 3 : 1; 2, 4 \rangle$	1
$\langle 0; 2, 3 : 3; 4, \{7, 8, 9\} \rangle$	3
$\langle 0; \{2, 3, 4\}, \{4, 5, 6\} : 3; \{2, 3\}, 9 \rangle$	18
$\langle 0; \{2, 3, 4\}, \{4, 5, 6\} : 5; \{7, 8, 9\}, \{9, 10, 11\} \rangle$	81

For $(\alpha, \beta) = (0, 2)$:

<i>Recurrence</i>	<i>number</i>
$\langle 0; \{1, 2\}, \{2, 3\} : 1; 1, 4 \rangle$	4
$\langle 0; \{1, 2\}, \{2, 3\} : 2; \{3, 4\}, \{4, 5\} \rangle$	16
$\langle 0; \{3, 4\}, \{4, 5\} : 4; 3, 10 \rangle$	4
$\langle 0; \{3, 4\}, \{4, 5\} : 6; \{9, 10\}, \{10, 11\} \rangle$	16

For $(\alpha, \beta) = (2, 1)$:

<i>Recurrence</i>	<i>number</i>
$\langle 0; 1, 1 : 1; 2, 2 \rangle$	1
$\langle 0; \{1, 2\}, \{2, 3\} : 1; 1, 2 \rangle$	4
$\langle 0; \{1, 2\}, \{2, 3\} : 2; 1, \{5, 6\} \rangle$	8
$\langle 0; \{1, 2\}, \{2, 3\} : 2; 2, \{4, 5\} \rangle$	8
$\langle 0; \{1, 2\}, \{2, 3\} : 3; \{4, 5\}, \{5, 6\} \rangle$	16

When $\beta = 0$, the situation is very different. We have the following result that holds for recursions of the form (1.1) with arbitrary arity k and any values of the parameters p_i :

Theorem 5.1. *For all $\alpha > 0$, there are either no $(\alpha, 0)$ -Conolly meta-Fibonacci recursions of the form (1.1), or there are infinitely many.*

Proof. Suppose that for some given α and set of parameters the recursion

$$(5.1) \quad R(n) = \sum_{i=1}^k R \left(n - s_i - \sum_{j=1}^{p_i} R(n - a_{ij}) \right)$$

together with c initial values has an $(\alpha, 0)$ -Conolly solution. Since the $(\alpha, 0)$ -Conolly solution sequence assumes the value of each positive integer α times in order, it must be the sequence $\lceil n/\alpha \rceil$. Thus, $R(n) = \lceil n/\alpha \rceil$ satisfies (5.1), the c initial values must be the first c values of the sequence $\lceil n/\alpha \rceil$, and for $n > c$ we know that the arguments $n - a_{ij} > 0$ and $n - s_i - \sum_{j=1}^{p_i} \lceil \frac{n-a_{ij}}{\alpha} \rceil > 0$.

Now we define a new, related recurrence with an $(\alpha, 0)$ -Conolly solution as follows: Set

$$(5.2) \quad P(n) = \sum_{i=1}^k P \left(n - s_i - p_i - \sum_{j=1}^{p_i} P(n - a_{ij} - \alpha) \right),$$

with $c + \alpha$ initial values which we take to be the first $c + \alpha$ values of the sequence $\lceil n/\alpha \rceil$. Observe that from the discussion in the paragraph above $P(n - a_{ij} - \alpha)$ is well-defined. We show below that in fact for $n > c + \alpha$ all the arguments of P on the right hand side of (5.2) are positive so the terms are well-defined.

We now show by induction that for all n , $P(n) = \lceil n/\alpha \rceil$. By our assumption for the initial conditions we have the base case. Assume that our hypothesis is true up to $n - 1$. Then

$$\begin{aligned} & \sum_{i=1}^k P \left(n - s_i - p_i - \sum_{j=1}^{p_i} P(n - a_{ij} - \alpha) \right) = \sum_{i=1}^k P \left(n - s_i - p_i - \sum_{j=1}^{p_i} (P(n - a_{ij}) - 1) \right) \\ & = \sum_{i=1}^k P \left(n - s_i - p_i + p_i - \sum_{j=1}^{p_i} P(n - a_{ij}) \right) = \sum_{i=1}^k P \left(n - s_i - \sum_{j=1}^{p_i} P(n - a_{ij}) \right) = \left\lceil \frac{n}{\alpha} \right\rceil. \end{aligned}$$

The last equality holds by our assumption that (5.1) has solution sequence $\lceil n/\alpha \rceil$ and by the fact that $n > n - s_i - \sum_{j=1}^{p_i} P(n - a_{ij}) > 0$ since $P(n - a_{ij}) = \lceil \frac{n-a_{ij}}{\alpha} \rceil$. This shows that $\lceil n/\alpha \rceil$ solves (5.2).

This proves that $P(n)$ has an $(\alpha, 0)$ -Conolly solution. We could repeat this process, replacing $R(n)$ with $P(n)$, and hence construct infinitely many different recursions with an $(\alpha, 0)$ -Conolly solution, thereby proving the desired result. \square

For example, in [4] it is shown that $\langle 0; 1 : 2; 3 \rangle$ is $(2, 0)$ -Conolly; by the above theorem we deduce that so too are $\langle 1; 3 : 3; 5 \rangle$, $\langle 2; 5 : 4; 7 \rangle$ and in general, $\langle x; 2x + 1 : x + 2; 2x + 3 \rangle$ for any $x \geq 0$.

We are now better equipped to return to the case $(\alpha, \beta) = (4, 0)$, the only (α, β) pair not yet covered in our discussion of 2-ary recursions with $p = 2$. Applying Theorems 4.1 and 5.1 we conclude that there are an infinite number of $(4, 0)$ -Conolly recursions.

It is evident that we can use the same reasoning as above for $p = 2$ to show that there are an infinite number of $(2p, 0)$ -Conolly 2-ary recursions for any p . In Theorem 5.2 following, we substantially improve this result by providing a complete list of all such recursions.

Central to Theorem 5.2 is the observation that since the $(\alpha, 0)$ -Conolly sequences have (by definition) constant frequency sequences with value α , they are equal to $\lceil \frac{n}{\alpha} \rceil$. By Corollary 2.1, if H is of the form (1.3) with an $(\alpha, 0)$ -Conolly solution, then $\alpha = 2p$. In what follows we provide necessary and sufficient conditions for the sequence $\lceil \frac{n}{2p} \rceil$ to be the solution of H .

For technical reasons, we need to distinguish between the property that a meta-Fibonacci recursion $A(n)$ with given initial conditions generates $B(n)$ as its (unique) solution sequence via a recursive calculation, and the property that the sequence $B(n)$ *formally satisfies* the recursion $A(n)$, by which we mean that for all n , $B(n)$ satisfies the equation that defines $A(n)$. An example makes this distinction clearer: consider the recursion $R = \langle -1; -1 : 2; 3 \rangle$. R is formally satisfied by the sequence $\lceil \frac{n}{2} \rceil$, but $\lceil \frac{n}{2} \rceil$ is not the solution sequence to R (for any set of initial conditions) because the recursion $R(n) = R(n+1 - R(n+1)) - R(n-2 - R(n-3))$ will always require that we know the term $R(n+1)$ to calculate $R(n)$.

Note that in the example above the recursion R has some negative parameters, a situation that we don't normally permit. As we will see in Corollary 5.1 no such example is possible without negative parameters.

Our strategy for classifying $(\alpha, 0)$ -Conolly sequences takes advantage of this distinction. We first show that the $(\alpha, 0)$ -Conolly sequence formally satisfies a 2-ary recursion if and only if that recursion's parameters meet a certain set of conditions described in Theorem 5.2 below. Then we show that formal satisfaction is equivalent to generating the sequence as the solution to the recurrence so long as we provide sufficiently many initial conditions that match the ceiling function.

For any integer z , let $z^{(q)}$ and $z^{(r)}$ be the quotient and remainder mod $2p$, so that $z = 2pz^{(q)} + z^{(r)}$ with $0 \leq z^{(r)} < 2p$.

Theorem 5.2. *The 2-ary, order p meta-Fibonacci recurrence relation*

$$H(n) = H\left(n - s - \sum_{i=1}^p H(n - a_i)\right) + H\left(n - t - \sum_{i=1}^p H(n - b_i)\right)$$

is formally satisfied by the sequence $\lceil \frac{n}{2p} \rceil$ for all integers n if and only if the parameters satisfy

- (1) *For each integer j in $\{0, 1, \dots, p-1\}$, at most j of the $a_i^{(r)}$ s satisfy $a_i^{(r)} \leq j$ and at most j of them satisfy $a_i^{(r)} \geq 2p - j$.*
- (2) *For each integer j in $\{0, 1, \dots, p-1\}$, at most j of the $b_i^{(r)}$ s satisfy $b_i^{(r)} \leq j$ and at most j of them satisfy $b_i^{(r)} \geq 2p - j$.*

- (3) *There exists an integer d such that either $-s + \sum_{i=1}^p a_i^{(q)} = 2pd$ and $-t + \sum_{i=1}^p b_i^{(q)} = -2pd - p$, or $-s + \sum_{i=1}^p a_i^{(q)} = -2pd - p$ and $-t + \sum_{i=1}^p b_i^{(q)} = 2pd$.*

Note that conditions (1), (2) and (3) are invariant under the transformation from R to P presented in Theorem 5.1 above, as well as under related transformations.

Proof. Clearly $H(n)$ is formally satisfied by $\left\lfloor \frac{n}{2p} \right\rfloor$ if and only if

$$\left\lfloor \frac{n}{2p} \right\rfloor = \left\lfloor \frac{n - s - \sum_{i=1}^p \left\lfloor \frac{n-a_i}{2p} \right\rfloor}{2p} \right\rfloor + \left\lfloor \frac{n - t - \sum_{i=1}^p \left\lfloor \frac{n-b_i}{2p} \right\rfloor}{2p} \right\rfloor.$$

Let $h(n) = h_1(n) + h_2(n)$ where

$$h_1(n) = \left\lfloor \frac{n - s - \sum_{i=1}^p \left\lfloor \frac{n-a_i}{2p} \right\rfloor}{2p} \right\rfloor,$$

$$\text{and } h_2(n) = \left\lfloor \frac{n - t - \sum_{i=1}^p \left\lfloor \frac{n-b_i}{2p} \right\rfloor}{2p} \right\rfloor.$$

Our strategy is to show that if $h(n) = \left\lfloor \frac{n}{2p} \right\rfloor$, then, up to interchanging h_1 and h_2 , $h_1(n) = \left\lfloor \frac{n}{4p} \right\rfloor + d$ and $h_2(n) = \left\lfloor \frac{n-2p}{4p} \right\rfloor - d$ which forces the stated conditions on the parameters. This is done in several steps, outlined as follows.

Lemma 5.1 shows that the successive differences of both $h_1(n)$ and $h_2(n)$ are always either -1, 0 or 1, regardless of the values of the parameters s, t, a_i and b_i . We assume that $h(n) = \left\lfloor \frac{n}{2p} \right\rfloor$, which is slowly growing, and in particular, $h(n+1) - h(n) = 1$ if and only if $n = 2p\mu$ for some integer μ . This, together with Lemma 5.1 implies that for a given integer μ , exactly one of h_1 or h_2 satisfies $h_i(2p\mu+1) - h_i(2p\mu) = 1$.

Lemma 5.2 and Lemma 5.3 impose conditions on $h_1(n)$ for certain intervals, while Lemmas 5.4 and 5.5 do the same for $h_2(n)$ in complementary intervals (see Figure 7). We prove that $h_1(n) = \left\lfloor \frac{n}{4p} \right\rfloor + d$ in the intervals where the remainder of n modulo $4p$ lies in the range $(p, 3p]$, which for convenience we write as $p < n \pmod{4p} \leq 3p$. We will show that in these intervals $h_1(n)$ is constant but $h(n)$ is not. Similarly, Lemmas 5.4 and 5.5 prove that $h_2(n) = \left\lfloor \frac{n-2p}{4p} \right\rfloor - d$ for the complementary intervals, $-p < n \pmod{4p} \leq p$, in which $h_2(n)$ is constant but $h(n)$ is not.

Thus for each interval, we know the values of $h(n)$ and exactly one of $h_1(n)$ or $h_2(n)$. Using $h(n) = h_1(n) + h_2(n)$ we are able to determine that $h_1(n) = \left\lfloor \frac{n}{4p} \right\rfloor + d$ and $h_2(n) = \left\lfloor \frac{n-2p}{4p} \right\rfloor - d$, for all n . This imposes the stated conditions on the parameters s, t, a_i and b_i . The converse will also become evident. That is, if we assume the parameters s, t, a_i and b_i satisfy conditions (1), (2) and (3), then $H(n)$ is satisfied by $\left\lfloor \frac{n}{2p} \right\rfloor$.

Lemma 5.1. *For all n , $|h_i(n+1) - h_i(n)| \leq 1$.*

Proof. We show that the numerator in h_1 does not change too quickly, which ensures that h_1 has successive differences that are small in magnitude.

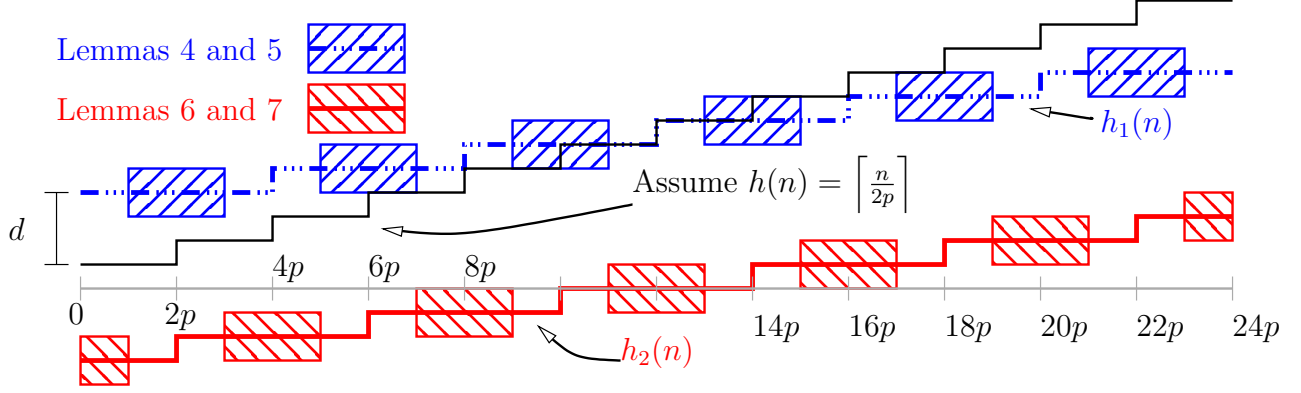


FIGURE 7. A visual representation of the proof of Theorem 5.2

Observe that $-1 \leq \left\lceil \frac{n-a_i}{2p} \right\rceil - \left\lceil \frac{n+1-a_i}{2p} \right\rceil \leq 0$, so that

$$-p \leq \sum_{i=1}^p \left(\left\lceil \frac{n-a_i}{2p} \right\rceil - \left\lceil \frac{n+1-a_i}{2p} \right\rceil \right) \leq 0.$$

Subtracting the numerators for successive arguments of h_1 we get

$$\begin{aligned} & \left| n+1-s - \sum_{i=1}^p \left\lceil \frac{n+1-a_i}{2p} \right\rceil - n+s + \sum_{i=1}^p \left\lceil \frac{n-a_i}{2p} \right\rceil \right| \\ &= \left| 1 + \sum_{i=1}^p \left(\left\lceil \frac{n-a_i}{2p} \right\rceil - \left\lceil \frac{n+1-a_i}{2p} \right\rceil \right) \right| < p < 2p. \end{aligned}$$

Thus $|h_1(n+1) - h_1(n)| \leq 1$. The proof for h_2 is similar, just replace s with t and a_i with b_i . \square

If $h(n+1) - h(n) = 1$, then necessarily $h_1(n+1) - h_1(n) > 0$ or $h_2(n+1) - h_2(n) > 0$. By Lemma 5.1 either $h_1(n+1) - h_1(n) = 1$ and $h_2(n+1) - h_2(n) = 0$ or vice versa. Without loss of generality, since we can interchange h_1 and h_2 , we assume that

$$(5.3) \quad h_1(1) - h_1(0) = 1 \text{ and } h_2(1) - h_2(0) = 0.$$

It is helpful to expand the numerator of $h_1(n)$ using the notation $z^{(q)}$ and $z^{(r)}$ for the quotient and remainder modulo $2p$ introduced above:

$$\begin{aligned}
 h_1(n) &= \left\lfloor \frac{2pn^{(q)} + n^{(r)} - s - \sum_{i=1}^p \left\lfloor \frac{2pn^{(q)} + n^{(r)} - 2pa_i^{(q)} - a_i^{(r)}}{2p} \right\rfloor}{2p} \right\rfloor \\
 &= \left\lfloor \frac{2pn^{(q)} + n^{(r)} - s - \sum_{i=1}^p \left(n^{(q)} - a_i^{(q)} + \left\lfloor \frac{n^{(r)} - a_i^{(r)}}{2p} \right\rfloor \right)}{2p} \right\rfloor \\
 (5.4) \quad &= \left\lfloor \frac{pn^{(q)} + n^{(r)} - s + \sum_{i=1}^p a_i^{(q)} - \sum_{i=1}^p \left\lfloor \frac{n^{(r)} - a_i^{(r)}}{2p} \right\rfloor}{2p} \right\rfloor.
 \end{aligned}$$

A similar expression holds for h_2 , where we replace s with t and a_i with b_i .

Observe that the sum $\sum_{i=1}^p \left\lfloor \frac{n^{(r)} - a_i^{(r)}}{2p} \right\rfloor$ in (5.4) is quite tame. In particular, $0 \leq n^{(r)} < 2p$ and $0 \leq a_i^{(r)} < 2p$ so that $\left\lfloor \frac{n^{(r)} - a_i^{(r)}}{2p} \right\rfloor$ is either 0 or 1, and is equal to 1 if and only if $n^{(r)}$ is strictly greater than $a_i^{(r)}$. Using the standard Iversonian notation, we write

$$\left\lfloor \frac{n^{(r)} - a_i^{(r)}}{2p} \right\rfloor = \llbracket n^{(r)} > a_i^{(r)} \rrbracket.$$

Using this notation we have the inequality $0 \leq \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket \leq p$, which is used extensively in the lemmas below to describe the behavior of h_i for certain intervals. We can combine the Iversonian notation with (5.4) to rewrite:

$$(5.5) \quad h_1(n) = \left\lfloor \frac{pn^{(q)} + n^{(r)} - s + \sum_{i=1}^p a_i^{(q)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rfloor.$$

Let $d = h_1(0)$, so that $h_2(0) = h(0) - h_1(0) = -d$. By (5.3) $h_1(1) = d+1$ while $h_2(1) = -d$.

Lemma 5.2. *For each i in $\{1, 2, \dots, p\}$, $a_i^{(r)} \neq 0$. Further, $-s + \sum_{i=1}^p a_i^{(q)} = 2pd$.*

Proof. We compare numerators in $h_1(0)$ and $h_1(1)$ using the form derived in (5.5). Notice that $\llbracket 0 > a_i^{(r)} \rrbracket = 0$ for every i , so that

$$h_1(0) = \left\lfloor \frac{-s + \sum_{i=1}^p a_i^{(q)}}{2p} \right\rfloor.$$

Since $h_1(1) - h_1(0) = 1$ and

$$h_1(1) = \left\lfloor \frac{1 - s + \sum_{i=1}^p a_i^{(q)} - \sum_{i=1}^p \llbracket 1 > a_i^{(r)} \rrbracket}{2p} \right\rfloor,$$

the numerator of $h_1(1)$ must be greater than the numerator of $h_1(0)$, that is,

$$-s + \sum_{i=1}^p a_i^{(q)} < 1 - s + \sum_{i=1}^p a_i^{(q)} - \sum_{i=1}^p \llbracket 1 > a_i^{(r)} \rrbracket,$$

which simplifies to

$$1 - \sum_{i=1}^p \llbracket 1 > a_i^{(r)} \rrbracket > 0.$$

It follows that for each $1 \leq i \leq p$ we have $\llbracket 1 > a_i^{(r)} \rrbracket = 0$ so $a_i^{(r)} \geq 1$. Thus, $h_1(1) = \left\lceil \frac{1-s+\sum_{i=1}^p a_i^{(q)}}{2p} \right\rceil = d+1$ and $h_1(0) = \left\lceil \frac{-s+\sum_{i=1}^p a_i^{(q)}}{2p} \right\rceil = d$ which implies that $-s + \sum_{i=1}^p a_i^{(q)} = 2pd$, as required. \square

We can now characterize the behavior of $h_1(n)$ whenever $p < n \pmod{4p} \leq 3p$.

Lemma 5.3. *With d defined above,*

$$h_1(n) = \left\lceil \frac{n}{4p} \right\rceil + d$$

whenever $p < n \pmod{4p} \leq 3p$.

Proof. We rewrite (5.5) using $-s + \sum_{i=1}^p a_i^{(q)} = 2pd$ from Lemma 5.2:

$$\begin{aligned} h_1(n) &= \left\lceil \frac{pn^{(q)} + n^{(r)} + 2pd - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil \\ &= d + \left\lceil \frac{pn^{(q)} + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil. \end{aligned}$$

To complete the proof of the lemma, we check that the ceiling function above is in fact $\left\lceil \frac{n}{4p} \right\rceil$ as desired. We separate into three cases:

Case 1: $p < n \pmod{4p} < 2p$. Observe that in this case, $n^{(q)}$ is even, so

$$\begin{aligned} \left\lceil \frac{pn^{(q)} + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil &= \frac{n^{(q)}}{2} + \left\lceil \frac{n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil \\ &= \frac{n - n^{(r)}}{4p} + \left\lceil \frac{n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil. \end{aligned}$$

Since $2p > n^{(r)} > p$, we have that $\frac{n-n^{(r)}}{4p} = \left\lceil \frac{n}{4p} \right\rceil - 1$, so we simply need to note that

$$\left\lceil \frac{n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil = 1 \text{ because } 2p > n^{(r)} > p \text{ and } \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket \leq p.$$

Case 2: $n \pmod{4p} = 2p$. This means $n^{(r)} = 0$, so

$$\left\lceil \frac{pn^{(q)} + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil = \left\lceil \frac{pn^{(q)}}{2p} \right\rceil = \left\lceil \frac{2pn^{(q)}}{4p} \right\rceil = \left\lceil \frac{n}{4p} \right\rceil.$$

Case 3: $2p < n \pmod{4p} \leq 3p$. Observe that in this case $n^{(q)}$ is odd. Hence

$$\begin{aligned} \left\lceil \frac{pn^{(q)} + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil &= \left\lceil \frac{pn^{(q)} + p - p + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil \\ &= \frac{n^{(q)} + 1}{2} + \left\lceil \frac{-p + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil. \end{aligned}$$

Since $0 < n^{(r)} \leq p$, we have $-2p < -p + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket \leq 0$, so that

$$\left\lceil \frac{-p + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil = 0.$$

Lastly, since $p \geq n^{(r)} > 0$, it is immediate that $\frac{n^{(q)}+1}{2} = \left\lceil \frac{n}{4p} \right\rceil$. \square

By Lemma 5.3 we know that $h_1(2p+1) - h_1(2p) = 0$. Since $h(2p+1) - h(2p) = 1$, we have that $h_2(2p+1) - h_2(2p) = 1$. Just as we used our assumption that $h_1(1) - h_1(0) = 1$ in the above two lemmas, we apply this crucial fact to prove the corresponding two lemmas for h_2 that follow. The technical details of the proofs are similar to those of the two preceding lemmas.

Lemma 5.4. *For each i in $\{1, 2, \dots, p\}$, $b_i^{(r)} \neq 0$. Further, with d as defined above, $-t + \sum_{i=1}^p b_i^{(q)} = -2pd - p$.*

Proof. Recall that if $n = 2p$, then $n^{(q)} = 1$ and $n^{(r)} = 0$. Following the proof of Lemma 5.2 we compare the numerators of $h_2(2p)$ and $h_2(2p+1)$. Using (5.5) and $\llbracket 0 > b_i^{(r)} \rrbracket = 0$, we have

$$h_2(2p) = \left\lceil \frac{p - t + \sum_{i=1}^p b_i^{(q)}}{2p} \right\rceil.$$

Since $h_2(2p+1) - h_2(2p) = 1$ and

$$h_2(2p+1) = \left\lceil \frac{p+1 - t + \sum_{i=1}^p b_i^{(q)} - \sum_{i=1}^p \llbracket 1 > b_i^{(r)} \rrbracket}{2p} \right\rceil,$$

the numerator of $h_2(2p+1)$ must be greater than that of $h_2(2p)$. This inequality,

$$p - t + \sum_{i=1}^p b_i^{(q)} < p + 1 - t + \sum_{i=1}^p b_i^{(q)} - \sum_{i=1}^p \llbracket 1 > b_i^{(r)} \rrbracket$$

simplifies to

$$1 - \sum_{i=1}^p \llbracket 1 > b_i^{(r)} \rrbracket > 0,$$

implying that $\llbracket 1 > b_i^{(r)} \rrbracket = 0$, so $b_i^{(r)} \geq 1$ for each $1 \leq i \leq p$. Thus,

$$h_2(2p+1) = \left\lceil \frac{p+1-t+\sum_{i=1}^p b_i^{(q)}}{2p} \right\rceil = -d + 1 \text{ and } h_2(2p) = \left\lceil \frac{p-t+\sum_{i=1}^p b_i^{(q)}}{2p} \right\rceil = -d \text{ which implies}$$

that $p - t + \sum_{i=1}^p b_i^{(q)} = -2pd$. \square

Note that Lemmas 5.4 and 5.2 establish condition (3).

We now prove the analogue to Lemma 5.3 for h_2 that we promised above.

Lemma 5.5. *Whenever $-p < n \pmod{4p} \leq p$, then $h_2(n) = \left\lceil \frac{n-2p}{4p} \right\rceil - d$.*

Proof. Let $g(n) = h_2(n+2p)$. We prove that $g(n) = \left\lceil \frac{n}{4p} \right\rceil - d$ whenever $p < n \pmod{4p} \leq 3p$. Using the analogue to (5.5) for h_2 and simplifying we get

$$h_2(n+2p) = \left\lceil \frac{2p(n^{(q)}+1) + n^{(r)} - t - \sum_{i=1}^p \left\lceil \frac{2p(n^{(q)}+1) + n^{(r)} - 2pb_i^{(q)} - b_i^{(r)}}{2p} \right\rceil}{2p} \right\rceil$$

$$h_2(n+2p) = \left\lceil \frac{pn^{(q)} + p + n^{(r)} - t + \sum_{i=1}^p b_i^{(q)} - \sum_{i=1}^p \llbracket n^{(r)} > b_i^{(r)} \rrbracket}{2p} \right\rceil$$

Recall from Lemma 5.4 that $p - t + \sum_{i=1}^p b_i^{(q)} = -2pd$. Thus, the remainder of the proof follows closely that for $h_1(n)$ in Lemma 5.3. Therefore $g(n) = h_2(n+2p) = \left\lceil \frac{n}{4p} \right\rceil - d$ so that $h_2(n) = \left\lceil \frac{n-2p}{4p} \right\rceil - d$ for $-p < n \pmod{4p} \leq p$. \square

Using the fact that h_1 and h_2 are constant for complementary ranges of n and that they are related by $h(n) = h_1(n) + h_2(n)$ we are able to characterize their behavior for all n .

Lemma 5.6. *For all n , $h_1(n) = \left\lceil \frac{n}{4p} \right\rceil + d$ and $h_2(n) = \left\lceil \frac{n-2p}{4p} \right\rceil - d$.*

Proof. It is easy to see that for all n the ceiling function satisfies

$$h(n) = \left\lceil \frac{n}{2p} \right\rceil = \left(\left\lceil \frac{n}{4p} \right\rceil + d \right) + \left(\left\lceil \frac{n-2p}{4p} \right\rceil - d \right).$$

When $p < n \pmod{4p} \leq 3p$ we have $h_1(n) = \left\lceil \frac{n}{4p} \right\rceil + d$ by Lemma 5.3; this implies that $h_2(n) = \left\lceil \frac{n-2p}{4p} \right\rceil - d$ in this range. When $-p < n \pmod{4p} \leq p$ we have $h_2(n) = \left\lceil \frac{n-2p}{4p} \right\rceil - d$ by Lemma 5.5; this implies that $h_1(n) = \left\lceil \frac{n}{4p} \right\rceil + d$ in this range. \square

Now that we have proven that for all n , $h_1(n) = \left\lceil \frac{n}{4p} \right\rceil + d$, we can show that condition (1) on the $a_i^{(r)}$ is necessary. Recall that in the proof of Lemma 5.3, we showed for all n , $h_1(n) = d + \left\lceil \frac{pn^{(q)} + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil$. Therefore we have that for all n ,

$$(5.6) \quad \left\lceil \frac{n}{4p} \right\rceil = \left\lceil \frac{pn^{(q)} + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil.$$

Our proof is driven by this ceiling equality. In fact, we only need to use n in the range $0 < n < 4p$ to get the desired results; the two segments $0 < n < 2p$ and $2p < n < 4p$ will force the two different parts of condition (1).

First, consider n in the range $0 < n < 2p$. In this case $n^{(r)} = n$ and $n^{(q)} = 0$. So (5.6) implies $\left\lceil \frac{n - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil = 1$, so $0 < n - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket \leq 2p$. From the left inequality we get $\sum_{i=1}^p \llbracket n > a_i^{(r)} \rrbracket < n$. It is easy to see how this implies the first part of condition (1): for each integer j in $\{0, 1, \dots, p-1\}$, let $j+1 = n$. Then by the preceding inequality $\sum_{i=1}^p \llbracket j \geq a_i^{(r)} \rrbracket \leq j$. That is, for each integer j in $\{0, 1, \dots, p-1\}$, at most j of the $a_i^{(r)}$ s satisfy $a_i^{(r)} \leq j$. Therefore this condition is necessary for Equation (5.6) to be satisfied for n in the range $0 < n \pmod{4p} < 2p$.

Next, consider $2p < n < 4p$. Here, $n^{(r)} = n - 2p$ and $n^{(q)} = 1$. Substituting these into (5.6) we get $\left\lceil \frac{p+n-2p-\sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil = 1$, which is equivalent to $0 < n-p-\sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket \leq 2p$. The right inequality implies $n-3p \leq \sum_{i=1}^p \llbracket n-2p > a_i^{(r)} \rrbracket$ so when $j = 4p-n$, we have $p-j \leq \sum_{i=1}^p \llbracket 2p-j > a_i^{(r)} \rrbracket$. That is, for each integer j in $\{1, 2, \dots, p-1\}$, at most j of the $a_i^{(r)}$ s satisfy $a_i^{(r)} \geq 2p-j$. Therefore, this condition is necessary for Equation (5.6) to be satisfied for n in the range $2p < n \pmod{4p} < 4p$. This completes the proof of the second part of condition (1) of the theorem.

If we substitute $m = n - 2p$ into $h_2(n) = \left\lceil \frac{n-2p}{4p} \right\rceil - d$ we can utilize the above argument to prove condition (2) of Theorem 5.2.

We have established the necessity of (1), (2) and (3). We now show sufficiency. Our strategy is simple: we will reverse our arguments to show that conditions (1),(2), and (3) imply that $h_1(n)$ and $h_2(n)$ are $\left\lceil \frac{n}{4p} \right\rceil + d$ and $\left\lceil \frac{n-2p}{4p} \right\rceil - d$. Assume that all three conditions hold. Without loss of generality, assume that $-s + \sum_{i=1}^p a_i^{(q)} = 2pd$. If not, switch the first and second summands in the recursion.

As shown previously, expanding the definition of $h_1(n)$ gives Equation (5.5), which for convenience we rewrite below:

$$h_1(n) = \left\lceil \frac{pn^{(q)} + n^{(r)} - s + \sum_{i=1}^p a_i^{(q)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil.$$

Substituting $2pd$ for $-s + \sum_{i=1}^p a_i^{(q)}$, then factoring out d and bringing d out of the ceiling function we get $h_1(n) = d + \left\lceil \frac{pn^{(q)} + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil$. A similar equation holds for $h_2(n)$.

Thus, if we can show that $\left\lceil \frac{pn^{(q)} + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil = \left\lceil \frac{n}{4p} \right\rceil$, with a corresponding equality for h_2 , we will be done. We show the details only for the required equality above for h_1 ; the approach in the second case is entirely similar.

First, note that if the desired equality above for h_1 holds for n , it also holds for $n + 4p$. This is because $\left\lceil \frac{p(n+4p)^{(q)} + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil = \left\lceil \frac{pn^{(q)} + 2p + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil = 1 + \left\lceil \frac{pn^{(q)} + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil = 1 + \left\lceil \frac{n}{4p} \right\rceil = \left\lceil \frac{n+4p}{4p} \right\rceil$. Thus, without loss of generality we consider only n with $1 \leq n \leq 4p$.

For all such n , $\left\lceil \frac{n}{4p} \right\rceil = 1$, so we need only prove that for $1 \leq n \leq 4p$, $\left\lceil \frac{pn^{(q)} + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket}{2p} \right\rceil =$

1. This is equivalent to proving that for $1 \leq n \leq 4p$, $1 \leq pn^{(q)} + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket \leq 2p$. We consider three cases.

Case 1: $1 \leq n < 2p$. In this case $n^{(q)} = 0$, $n^{(r)} = n$. We want $1 \leq n - \sum_{i=1}^p \llbracket n > a_i^{(r)} \rrbracket \leq 2p$. Since $\llbracket n > a_i^{(r)} \rrbracket = 1 - \llbracket n \leq a_i^{(r)} \rrbracket$, the prior inequality is equivalent to $1 \leq n - \sum_{i=1}^p (1 - \llbracket n \leq a_i^{(r)} \rrbracket) \leq 2p$, or $1 \leq n - p + \sum_{i=1}^p \llbracket n \leq a_i^{(r)} \rrbracket \leq 2p$, which we can rearrange as $1 + p - n \leq \sum_{i=1}^p \llbracket n \leq a_i^{(r)} \rrbracket \leq 3p - n$. Because $n < 2p$, the right-hand inequality is always true, and clearly the left-hand inequality can only possibly be false for $n \leq p$. By condition (1), at most $n - 1$ of the $a_i^{(r)}$ satisfy $a_i^{(r)} \leq n - 1$, which is equivalent to $a_i^{(r)} < n$. Therefore, at least $p - (n - 1)$ of the $a_i^{(r)}$ satisfy the negation, $n \leq a_i^{(r)}$, so $1 + p - n \leq \sum_{i=1}^p \llbracket n \leq a_i^{(r)} \rrbracket$.

Case 2: $n = 2p, 4p$. In this case, $n^{(q)} = 1$ or 2 , and $n^{(r)} = 0$, which makes $pn^{(q)} + n^{(r)} - \sum_{i=1}^p \llbracket n^{(r)} > a_i^{(r)} \rrbracket = p$ or $2p$, satisfying the required inequality.

Case 3: $2p < n < 4p$. In this case, $n^{(q)} = 1$ and $n^{(r)} = n - 2p$. Thus, the required inequality is $1 \leq p + n - 2p - \sum_{i=1}^p \llbracket n - 2p > a_i^{(r)} \rrbracket \leq 2p$. The left inequality is always true since $n > 2p$. So we need only show that $n - p - \sum_{i=1}^p \llbracket n - 2p > a_i^{(r)} \rrbracket \leq 2p$, which we may rewrite into $n - 3p \leq \sum_{i=1}^p \llbracket n - 2p > a_i^{(r)} \rrbracket$. Observe that $\llbracket n - 2p > a_i^{(r)} \rrbracket = 1 - \llbracket n - 2p \leq a_i^{(r)} \rrbracket$, so the required inequality becomes $n - 3p \leq \sum_{i=1}^p (1 - \llbracket n - 2p \leq a_i^{(r)} \rrbracket)$, or $4p - n \geq \sum_{i=1}^p \llbracket n - 2p \leq a_i^{(r)} \rrbracket$. If $n \leq 3p$, this inequality is obviously true, so we need only consider $3p < n < 4p$, in which case $0 \leq 4p - n < p$. By condition (1), there are at most $4p - n$ values of i with $a_i^{(r)} \geq 2p - (4p - n) = n - 2p$, which proves the desired inequality. \square

Theorem 5.2 gives a complete characterization of 2-ary meta-Fibonacci recurrences that are formally satisfied by the sequence $\left\lceil \frac{n}{\alpha} \right\rceil^6$. As discussed above, this is not the same as generating this sequence as the solution sequence from the recurrence, since in theory the recurrence relation might refer to future terms. The following corollary rules out this possibility:

Corollary 5.1. *If $H(n)$ satisfies conditions (1)–(3) in Theorem 5.2, then, given a sufficient number of initial values, $H(n)$ references only terms in $\{1, 2, \dots, n - 1\}$.*

Proof. By our assumption concerning the parameters in Equation (1.3), $a_i > 0$ and $b_i > 0$ and also s and t are non-negative. Thus $H(n - a_i)$ and $H(n - b_i)$ are not references to future terms. It is easy to see that $0 < n - s - \sum_{i=1}^p H(n - a_i) \leq n - 1$ because $\sum_{i=1}^p H(n - a_i)$ grows asymptotically like $n/2$. \square

We now present two special cases of Theorem 5.2 for $p = 1$ and $p = 2$, the first of which was given as Corollary 3.1. If $p = 1$ (writing $a_1 = a$ and $b_1 = b$) then conditions (1) and (2) of Theorem 5.2 say that a and b are odd while condition (3) says, up to the usual symmetry in the parameters, that for some integer d , $\lfloor \frac{a}{2} \rfloor - s = 2d$ and $\lfloor \frac{b}{2} \rfloor - t = -2d - 1$. If we add the equations in (3) and multiply by 2 we get $2 \lfloor \frac{a}{2} \rfloor - 2s + 2 \lfloor \frac{b}{2} \rfloor - 2t = -2$, from which we derive the result promised in Corollary 3.1, namely, $2(s + t) = a + b$, with a and b both odd.

If $p = 2$ then the conditions of Theorem 5.2 can be simplified as follows:

⁶Note that Theorem 5.2 actually shows formal satisfaction even when we do not require positivity of the parameters.

Corollary 5.2. *Let $H(n) = H(n-s-H(n-a)-H(n-b)) + H(n-t-H(n-c)-H(n-d))$ be an order 2 meta-Fibonacci recurrence relation. The sequence $\left\lceil \frac{n}{4} \right\rceil$ is a solution to $H(n)$ if and only if there is an odd integer k such that the following conditions are satisfied:*

- (i) $a + b \in \{4(s+k) - 1, 4(s+k), 4(s+k) + 1\}$,
- (ii) $c + d \in \{4(t-k) - 1, 4(t-k), 4(t-k) + 1\}$,
- (iii) $a, b, c, d \not\equiv 0 \pmod{4}$.

Proof. Suppose $\left\lceil \frac{n}{4} \right\rceil$ is a solution to $H(n)$. We apply Theorem 5.2 with $p = 2$. Condition (3) of Theorem 5.2 says, up to switching the roles of $\{a, b, s\}$ and $\{c, d, t\}$, that $\left\lceil \frac{a}{4} \right\rceil + \left\lceil \frac{b}{4} \right\rceil = s + 4e$ and $\left\lceil \frac{c}{4} \right\rceil + \left\lceil \frac{d}{4} \right\rceil = t - 4e - 2$. Multiplying both sides by 4 and adding $a^{(r)} + b^{(r)}$ gives $a + b = 4(s + 4e) + a^{(r)} + b^{(r)}$ while a similar operation gives $c + d = 4(t - 4e - 2) + c^{(r)} + d^{(r)}$. From conditions (1) and (2) we have (iii) and $a^{(r)} + b^{(r)} \in \{3, 4, 5\}$ and $c^{(r)} + d^{(r)} \in \{3, 4, 5\}$. So $a + b = 4(s + 4e + 1) + a^{(r)} + b^{(r)} - 4$, and $c + d = 4(t - 4e - 2 + 1) + c^{(r)} + d^{(r)} - 4$. Since $4e + 1 = -(-4e - 2 + 1)$ then either $\kappa = 4e + 1$ or $\kappa = -4e - 2 + 1 = -(4e + 1)$, so κ can be any odd integer.

The argument can be reversed so that (1), (2) and (3) follow from (i), (ii) and (iii). That is, assume κ is an odd integer. If $\kappa = 4e + 1$ for some integer e , then by (i), we have $a + b = 4(s + 4e) + E$ where $E \in \{3, 4, 5\}$. Subtracting $a^{(r)} + b^{(r)}$ and dividing by 4 gives $\left\lceil \frac{a}{4} \right\rceil + \left\lceil \frac{b}{4} \right\rceil = s + 4e + \frac{E - a^{(r)} - b^{(r)}}{4}$. But the left hand side is an integer, so $E - a^{(r)} - b^{(r)} \pmod{4} = 0$. Since $a, b \not\equiv 0 \pmod{4}$, we have $1 \leq a^{(r)} \leq 3$, and $1 \leq b^{(r)} \leq 3$, so $2 \leq a^{(r)} + b^{(r)} \leq 6$. Therefore, we must have $E = a^{(r)} + b^{(r)}$. This forbids $a^{(r)} = b^{(r)} = 1$ and $a^{(r)} = b^{(r)} = 3$. Combining this with condition (iii), we establish condition (1). A similar argument yields condition (2). By Theorem 5.2, therefore, $\left\lceil \frac{n}{4} \right\rceil$ is a solution to $H(n)$. \square

6. CONCLUDING REMARKS

We have an almost complete characterization of 2-ary order 1 Conolly-like recursions. The only missing component is a proof that $\langle 0; 1 : 1; 2 \rangle$ and $\langle 0; 2 : 3; 5 \rangle$ are the only two order 1 recursions satisfied by the Conolly sequence. To show this, we would like an analogue of Corollary 3.1 for $(0, 1)$ -Conolly recursions.

For 2-ary recursions of higher order we have shown that (α, β) -Conolly recursions exist for all permissible pairs (α, β) . Unlike the situation for order 1, we believe that for $\beta > 0$ there are many (α, β) -Conolly recursions. To expand on the existence result proved in Theorem 4.1 we would like an analogue to Theorem 5.2; a starting point would be to find a way to show that all of the 180 recursions listed in Conjecture 5.1 are indeed Conolly-like.

Nothing is currently known about the existence of Conolly-like recursions with arity higher than 2. An empirical investigation along the lines described in Section 2 likely would be a useful starting point.

Another direction for further work is that Theorem 5.2 seems adaptable to k -ary recurrence relations of order p and arbitrary α . A sequence $A(n)$ that satisfies $\left\lceil \frac{n}{\alpha} \right\rceil$ also satisfies the limit $\frac{A(n)}{n} \longrightarrow \frac{1}{\alpha}$. By Theorem 2.1, $\alpha = \frac{kp}{k-1}$ and since $\gcd(k, k-1) = 1$, we must have $p = q(k-1)$ for some q . Following the pattern of Theorem 5.2, one would want to use the fact that

$$(6.1) \quad \left\lceil \frac{n}{kq} \right\rceil = \left\lceil \frac{n}{k^2q} \right\rceil + \left\lceil \frac{n-kq}{k^2q} \right\rceil + \left\lceil \frac{n-2kq}{k^2q} \right\rceil + \cdots + \left\lceil \frac{n-(k-1)q}{k^2q} \right\rceil,$$

to show that the i th term on the right hand side corresponds to the i th term in the sum $H(n) = \sum_{i=1}^k h_i(n)$ for some k -ary recurrence relation H of order p . In this case one would define $m^{(a)}$ and $m^{(r)}$ to satisfy $m = kqm^{(a)} + m^{(r)}$ for $0 \leq m^{(r)} < kq$ so as to take advantage of the Iversonians used in Theorem 5.2. Particular attention should be paid to showing that the form in Equation (6.1) is necessary, given k and q .

Further consideration might be given to sequences of the form $\left\lceil \frac{rn}{q} \right\rceil$ with positive integers r and q such that $0 < r/q \leq 1$ and $\gcd(r, q) = 1$. These are only Conolly-like for $r = 1$, but nevertheless they have some interesting features. For example, this is a necessary and sufficient condition for a slow-growing sequence to have a periodic frequency function. These sequences will be the subject of a forthcoming publication.

Finally, it would be interesting to apply the tree technique to help identify recursions with solutions whose frequency functions are linear combinations of functions other than 1 and r_m . This will be the subject of a forthcoming publication.

REFERENCES

- [1] R.B.J.T. Allenby and R.C. Smith, *Some sequences resembling Hofstadter's* J. Korean Math. Soc. 40 (2003) 921–932.
- [2] Jean-Paul Allouche and Jeffrey Shallitt, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge University Press, 2003.
- [3] B. Balamohan, A. Kuznetsov, and S. Tanny, *On the Behavior of a Variant of Hofstadter's Q-Sequence*, Journal of Integer Sequences, Vol. 10 (2007), Article 07.7.1, electronic, 29 pages.
- [4] B. Balamohan, Z. Li, and S. Tanny, A combinatorial interpretation for certain relatives of the Conolly sequence, *J. of Integer Sequences* 11 (2008), Article 08.2.1.
- [5] Joseph Callaghan, John J. Chew III, Stephen M. Tanny, *On the Behavior of a Family of Meta-Fibonacci Sequences*, SIAM J. Discrete Math. 18(4): 794–824 (2005).
- [6] B. W. Conolly, Meta-Fibonacci sequences, in S. Vajda, ed., *Fibonacci & Lucas Numbers, and the Golden Section*, Wiley, New York, 1986, pp. 127–137.
- [7] Solomon W. Golomb, *Discrete chaos: sequences satisfying strange recursions*, preprint, undated.
- [8] J. Higham and S. Tanny, More well-behaved meta-Fibonacci sequences, *Congr. Numer.* 98 (1993), 3–17.
- [9] Douglas R. Hofstadter, *Godel, Escher, and Bach: An Eternal Golden Braid*, Basic Books, New York, 1979.
- [10] Abraham Isgur, David Reiss, and Stephen Tanny, Trees and meta-Fibonacci sequences, *Electron. J. of Combin.* 16 (2009), R129.
- [11] B. Jackson and F. Ruskey, Meta-Fibonacci sequences, binary trees and extremal compact codes, *Electron. J. of Combin.* 13 (2006), R26.
- [12] F. Ruskey and C. Deugau, The combinatorics of certain k -ary meta-Fibonacci sequences, *Journal of Integer Sequences*, Volume 12, Article 09.4.3, 36 pages, 2009.
- [13] N. J. A. Sloane, *Online Encyclopedia of Integer Sequences*, <http://www.research.att.com/~njas/sequences>.
- [14] T. Stoll, *On Hofstadter's married functions*, Fibonacci Quarterly 46/47 (2008/2009), 62–67.
- [15] S.M. Tanny, *A well-behaved cousin of the Hofstadter sequence*, Discrete Mathematics 105 (1992) 227–239.

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