These notes are based on material from the Digital Signal Processing Primer by Ken Steiglitz.

Sampling a phasor - Aliasing

The sound we hear travels as longitudinal waves of compression in the air, just like the standing waves in a tube. If we imagine a microphone diaphragm being pushed back and forth by an impringing wave front, we can represent the sound by a single real-valued function of time, say $x(t)$, which represents the displacement of the microphone's diaphragm from its resting position. To represent such a function digitally we need to discretize the real-valued time varitable t, called *sampling*; and we need to discretize the real-valued pressure variable $x(t)$, called *quantizing*. An analog-to-digital (ADC) converter performs both functions, producing a sequence of numbers representing successive samples of the sound pressure wave. A digital-to-analog (DAC) converter performs the reverse process.

There are issues we have to be aware when sampling and how it affects frequency components of a sound. Suppose we sample a simple sinusoidal signal. ADC converters take samples at regularly spaced time intervals. For example audio CDs are samples 44100 times a second. The terminology is that the sampling frequency or sampling rate is 44.1 kHz even if we are creating the sound from scratch in the computer. We will use f_s for the samping rate in Hz, ω_s for the sampling rate in radians per sec, and $T_s = 1/f_s$ for the interval between samples in seconds (the sampling period).

If the sampling rate is high compared to the frequency of the sinusoid, there is no problem. We get several samples to represent each cycle (period) of the sinusoid.

However, suppose we decrease the sampling rate, while keeping the frequency of the sinusoid constant. We get fewer and fewer samples per cycle and eventually that causes a problem. It is easier to understand what's happening if we return to our view of the sinusoid as the projection of a complex phasor.

Imagine a complex phasor rotating at a fixed frequency, and suppose that when we sample it, we paint a dot on the unit circle at the position of the phasor at the sample time. If we sample fast compared to the frequency of the phasor, the dots will be closely spaced, starting at the initial position of the phasor, and progreesing around the circle. We have an accurate

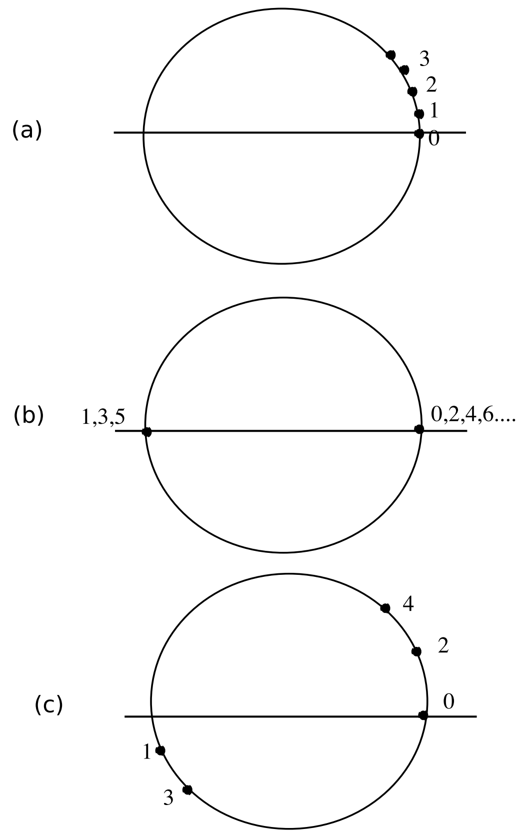

Figure 1: Sampling a phasor. (a) sampling rate high compared to the frequency of the phasor (b) the sampling rate is precisely half the frequency of the phasor (c) the sampling rate is slightly less than half the frequency of the phasor. In the last case the samples appear to move less than π in the clockwise (negative) direction.

representation of the phasor's frequency. See figure 1a.

Suppose we keep decreasing the sampling rate. The dots become more widely spaced around the circle until the situation shown in Figure 1b is reached. In this case the first sample is at point +1 in the plane and the second point is at point -1, the thid at +1, and so on. We know that the frequency of the sinusoid is now half the sampling rate, because we are taking two samples per revoluation of the phasor. We are stretched to the limit, however.

Let's see what happens if we sample at an even slower rate, so that the frequency of the phasor is a bit higher than half the sampling rate. The result is shown in Figure 1c . The problem now is that this result is indistinguishable from the result we would have obtained if the frequency of the phasor were a bit lower than half the sampling rate. Each successive dot can be thought of as rotated a little less than π radians in the negative direction. As far as projections on the real-axis are concerned it doesn't matter which way the phasor is rotating. This means that only frequencies below half the sampling rate will be accurately represented after sampling. This special frequency, half the sampling rate, is called the Nyquist frequency, after the American engineer Harry Nyquist (1889-1976).

To summarize: If we are sampling at the rate ω_s, we can't distinguish the differences between any frequency ω_0 and ω_0 plus any multiple of the sampling rate itself. As far as the real signal generated by the rotating phasor we also can't distinguish between the frequency ω_o and $-\omega_0$. Therefore all the following frequency are aliases of each other:

$$\pm\omega_0 + k\omega_s, \quad for\ \ all\ \ k\ \ \ integer \tag{1}$$

Homework

Aliasing can be observed in the world around you. Identify the source of the original signal and the sampling mechanism in the following situations:

- The hubcap of a car coming to a stop in a motion picture
- A TV news anchor squirming while wearing a tweed jacket
- A helicopter blade while the helicopter is starting up on a sunny day