These notes are based on material from the Digital Signal Processing Primer by Ken Steiglitz.

Adding sinusoids of the same frequency as phasors

We can think of the cosine function as a projection onto the x-axis of a point moving around the unit circle at constant speed. Actually the speed is simple ω radians per second, and it takes $\frac{2pi}{\omega}$ seconds to complete one revolution or period. Similarly for the sine wave and the y-axis projection. Figure 1 and 2 illustrates this. It is easier to think of a rotating clock-hand than some specially shaped curve. The position of the vector at the instant $t = 0$ tells us the relative phase of the sinusoid and the legth of the vector tells us the size or *magnitude* of the sinusoid.

Consider two sinusoids of the same frequency but with different amplitudes and phases. For a particular time instance they will correspond to two clock hands with different lengths separated by an angle. To add them the parallelogram law can be used (essentially we add their x-components and y-components separately). However we need to take into account that the vectors representing sinusoids are actually rotating. But if the frequencies of the sinusoids are the same, the vectors rotate at the same speed therefore their sum will also rotate at the same speed. It is as if the vectors and their sum were made out of steel and the joints of the parallelogram were welded together.

Complex Numbers

We will consider a geomertic interpretation of complex numbers as an elegant system for manipulating rotating vectors. Complex periodic sounds can be decomposed into a sum of sinusoidal components. Thinking of sinusoids as rotating vectors rather than signals going up and down provided us with easier insight. For example the property that adding two sinusoids of the same frequency with different amplitude and phases results in a sinusoid of the same frequency can be understood easily by thinking of rotating vectors but is not obvious when thinking of sinusoids as varying signals over time.

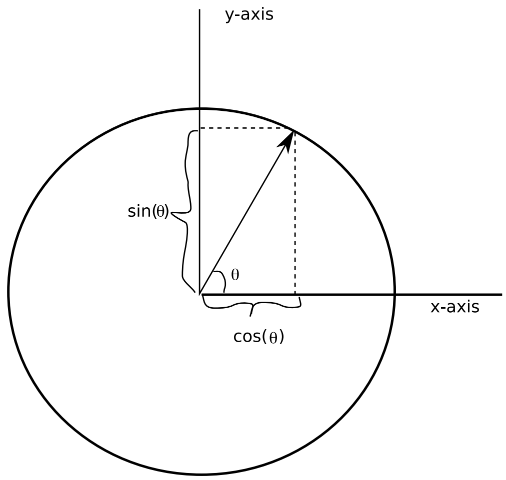

Figure 1: Definition of cosine and sine as projections from the unit circle to the x- and y-axes

The basic idea will be simple: to represent a vctor with an x-axis component x and a y-axis component y by the complex number $x + jy$ where x is called the *real* part and y is called the *imaginary* part. The key idea is that we will interpret multiplication with any complex number as a rotation operator. As a special case consider the complex number j (with real part equal to 0 and imaginary part equal to 1) as an operator that rotates by $+\frac{\pi}{2}$ (counterclockwise). Thus two successive multiplication by j of 1 bring us to the negative real axis, so $j^2 = -1$. This geometric viewpoint makes is clear that there is nothing mystical or imaginary about what might seem to be an impossible thing - a number whose square is -1.

A complex sinusoid is simply:

$$cos(\omega t) + jsin(\omega t) \tag{1}$$

Addition of complex numbers is identical with vector addition (we add the real parts and the imaginary parts). However the extra mileage we get from complex numbers arises from multiplication.

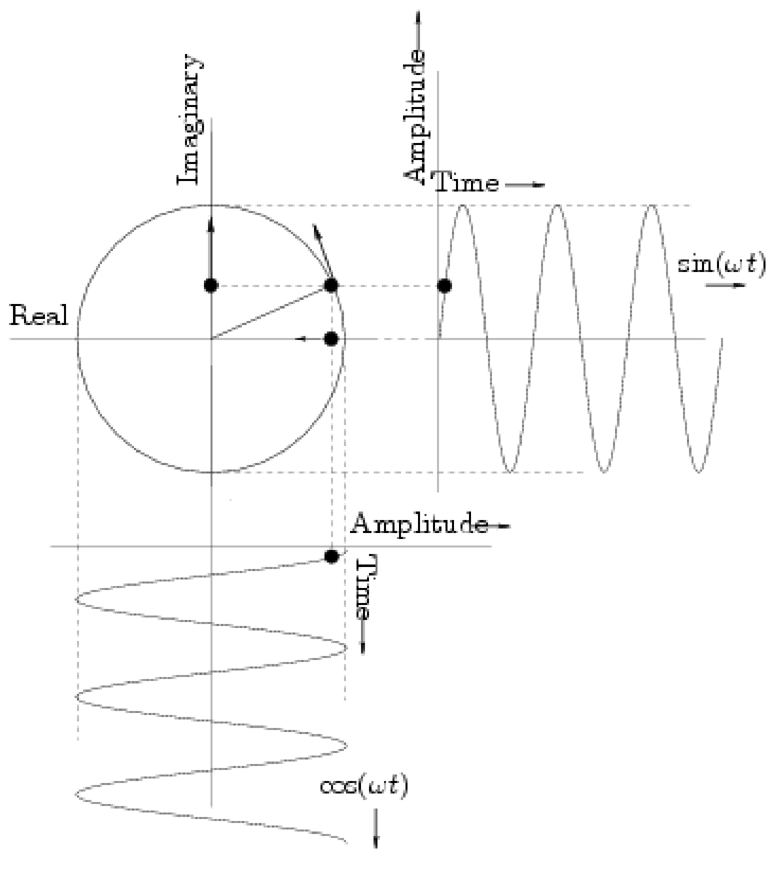

Figure 2: Cosine and sine waves considered as projections of a point moving around the unit circle at a constant speed

Multiplying Complex Numbers

To multiply two complex number we follow the rules of algebra blindly, using the usual distributive and commutative laws and replacing j^2 by -1. For example to multiply $a_1 + jb_1$ and $a_2 + jb_2$ we do:

$$(a_1 + jb_1) * (a_2 + jb_2) = (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1) \tag{2}$$

Notice that on a computer implementation there is no need to use j which is just a notational convenience so that we can use the standard multiplication rules we have learned in high school. In a computer implementation we just define the real value of the multiplication and the imaginary value directly.

The real beauty of this definition is revealed when we think of complex numbers as vectors in polar form. The length of the vector called magnitude and written $|z|$ is:

$$R = |z| = \sqrt{x^2 + y^2} \tag{3}$$

The angle it makes with the real axis is:

$$\theta = ARG(z) = arctan(y/x) \tag{4}$$

To go back we can use:

$$x = Rcos\theta \tag{5}$$

$$y = Rsin\theta \tag{6}$$

We would like complex number multiplication to be equivalent to real-number multiplication when the imaginary part is zero. This suggest that the magnitude of the product of two complex numbers should be equal to the product of their magnitude (at least for the case above).

Interpreting j as rotation by $\frac{\pi}{2}$ means that we want multiplication by j to add $\frac{\pi}{2}$ to the angle of any complex number but leave it's magnitude unchanged. This suggest that multiplication in general should result in adding the angle of the two complex numbers involved.

So we can try the following process to multiply two complex numbers in polar form: *Multiply the magnitudes and add the angles*. You can verify the algebra using the definition of multiplication above and converting from the polar representation as a homework exercise.

Euler's formula

Consider a fixed complex number, say W that represents a rotating vector frozen at some angle; W^2 represents the vector at twice that angle, W^3 at three times, and so forth. Not only that, but the vector W^p will represent a continuously roating vectors, where p is allowed to vary continuously over all possible real values, not just over integer values. This motivates the key insight of this section which is that the rotating vector representing a sinusoids can be thought of ass a single fixed complex number raised to progressively higher and higher powers.

We concentrate our attention on a rotating vector of unit magnitude. More precisely we consider the function:

$$E(\theta) = cos\theta + jsin\theta \tag{7}$$

which represents the vector at some arbitrary angle θ. From this we can find the derivative of $E(\theta)$ with respect to θ as:

$$\frac{dE(\theta)}{d\theta} = -sin\theta + jcos\theta \tag{8}$$

Looking at the equation above we notice that the effect of the differentiation was simply to multiply $cos\theta + j\sin theta$ by j. So the function $E(\theta)$ must follow the property:

$$\frac{dE(\theta)}{d\theta} = jE(\theta) \tag{9}$$

The only function that obays this simple law is the exponential function. Therefore:

$$E(\theta) = e^{j\theta} \tag{10}$$

This relation written out fully is:

$$cos\theta = jsin\theta = e^{j\theta} \tag{11}$$

and is called Euler's formula, after Swiss mathematician Leonhard Euler (1707-1783). It is one of the most remarkable formulas of mathematics. It fulfills the promise above that the rotating vector can be represented as a fixed complex number raised to higher and higher powers.

Euler's formula also ties together, in one compact embrace, the five best numbers in the universe, namely 0, 1, π, e, and j. To see this, just set $\theta = \pi$ and rearrange slightly (for aesthetic effect):

$$e^{j\pi} + 1 = 0 \tag{12}$$

Not only that but it also uses exactly once the three basic operations of addition, multiplication and exponentation - and the equality relation.

Homework

The phenomenon of beats arrises when we add two sinusoids with frequencies of vibration that are close but not identical. Create a plot in MATLAB showing this phenomenon. For example you can consider: $sin(\omega t) + 0.7 * sin((\omega + \delta)t)$, where $\omega = 0.3157$ radians per sec, and $\delta = 0.02$ radians per sec. Try to explain (to yourself) intuitively the phenomenon of beating using the phasor geometric viewpoint.