Write a $\lambda - calculus$ expression for the function *twice* that applies twice its first argument (a function) to it's second argument. Assume $square = \lambda x. * xx$ in lambda calculus. Show the steps in an applicative order and normal order reduction of the expression (twice (twice square)).

1 Applicative order (reduce argument then substitute

Applicative order means that for applications (F E) (F applied to E) E must be first reduced before it is substituted in the body of F. In our example that means that the inner (twice (square)) must be first be reduced before it is substituted in the body of the outer twice. Let's assume that E=(twice square). Then the first step we have to perform is to reduce E.

$$E = ((\lambda f.\lambda x.f(f\ x))square) \tag{1}$$

Using beta-reduction we have: (f is substituted in the body of the lambda abstraction with square)

$$E = (\lambda x.square(square\ x)) \tag{2}$$

Now we can evaluate (F E) by beta-reduction.

$$(F\ E) = ((\lambda f.\lambda x.f(f\ x))\ E) \tag{3}$$

And we get (using beta reduction):

$$(\lambda x.E(E\ x)) \tag{4}$$

Now the inner (E x) needs to be evaluated:

$$(E\ x) = ((\lambda x.square(square\ x))\ x) \tag{5}$$

Using beta reduction

$$(square(square\ x)) \tag{6}$$

So we have:

$$(\lambda x.(\lambda x.square(square(x)))(square(square(x)))) \tag{7}$$

After beta-reduction of the inner lambda we get:

$$(\lambda x.(square(square(square(squarex))))) \tag{8}$$

2 Normal Order (substitute argument then reduce)

In normal order when we have lambda-expressions of the form (F E) E is directly substituted in the body of F. Let's assume that E=(twice square). Then the first step is to substitute E directly into F.

$$(FE) = ((\lambda f.\lambda x.f(f\ x))\ E) \tag{9}$$

so we want:

$$(\lambda x.E(E\ x)) \tag{10}$$

and E is:

$$E = ((\lambda f.\lambda x.f(f\ x))square) \tag{11}$$

So by substitution we have:

$$(\lambda x.((\lambda f.\lambda x.f(f\ x))square)(((\lambda f.\lambda x.f(f\ x))square)\ x)) \tag{12}$$

There is nothing we can do about the outer lambda x so we reduce the first labmda f.

$$(\lambda x.(\lambda x.square(square\ x))(((\lambda f.\lambda x.f(f\ x))square)\ x)) \tag{13}$$

Now we can substitue the (E x) part in the place by applying the second lambda x:

$$(\lambda x.(square(square\ (((\lambda f.\lambda x.f(f\ x))square)\ x)))) \tag{14}$$

Similarly we can reduce:

$$(\lambda x.(square(square\ ((\lambda x.square(square\ x))\ x)))) \tag{15}$$

Which can be reduced:

$$(\lambda x.(square(square\ (square(square\ x))))) \tag{16}$$

No further reduction is possible