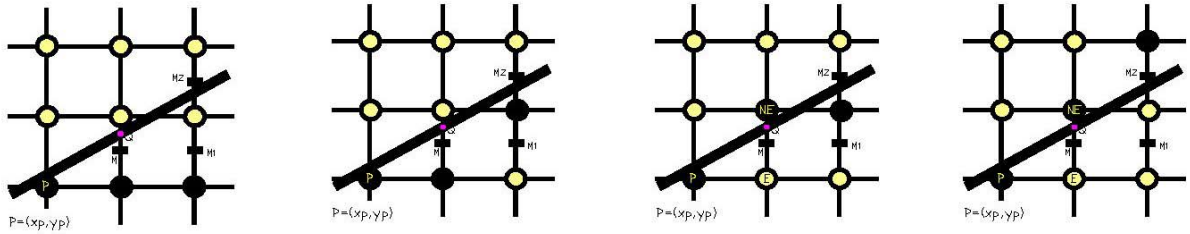
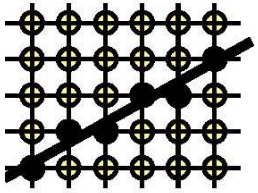
Computer Graphics

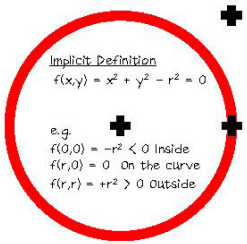


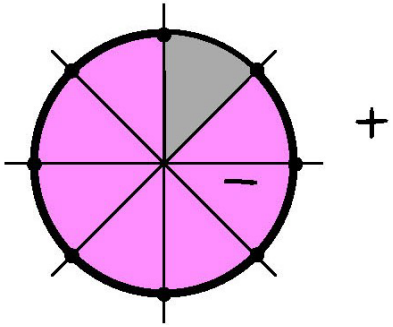
Pattern 1 Pattern 2 Pattern 3 Pattern 4



Scan Conversion

Lines & circles





by
Brian Wyvill

With thanks to

Prof. Rich Riesenfeld (University of Utah)

for some slides

Dr. Jack Bresenham

for some interesting discussions

Implicit Vs. Parametric

Implicit Definition

$f(x,y) = x^2 + y^2 - r^2 = 0$

e.g. $r = 1$

$f(0,0) = 0 + 0 - 1 < 0$ inside

$f(0,0) = 1 + 1 - 1 > 0$ outside

implies search space to find x,y to satisfy: $f(x,y) = 0$

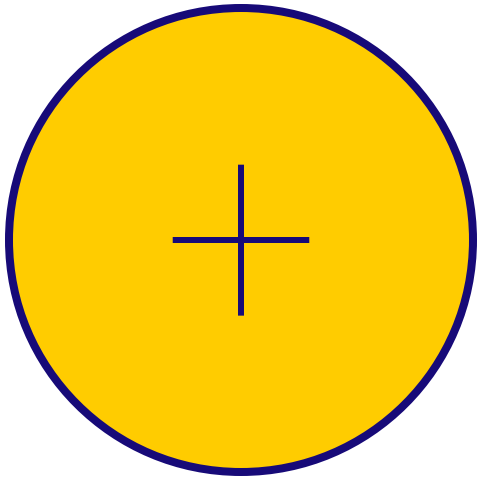
iso-surface: $f(x,y) - c = 0$

Parametric Definition

$x = r \sin(\alpha)$

$y = r \cos(\alpha)$

$0 \leqq \alpha \leqq 2\pi$



Line Characterisations

- **Explicit:** $y = mx + B$
- **Implicit:** $F(x, y) = ax + by + c = 0$
- **Constant slope:** $\dfrac{\Delta y}{\Delta x} = k$
- **Constant derivative:** $f'(x) = k$

Line Characterisations

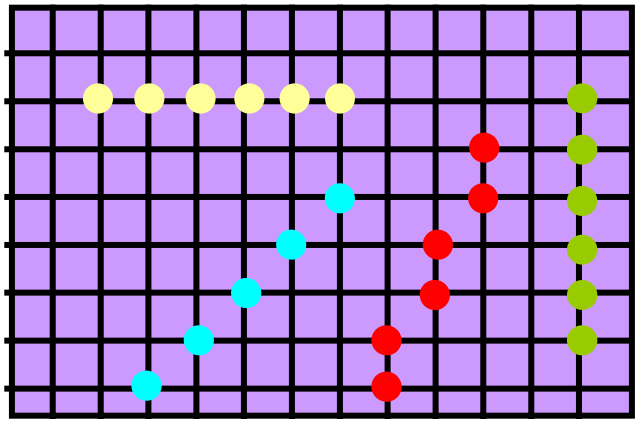
- **Parametric:** $$P(t) = (1-t)\,P_0 + t\,P_1$$

  where, $$P(0) = P_0\,; \quad P(1) = P_1$$

- **Intersection of 2 planes**
- **Shortest path between 2 points**
- ***Convex hull* of 2 discrete points**

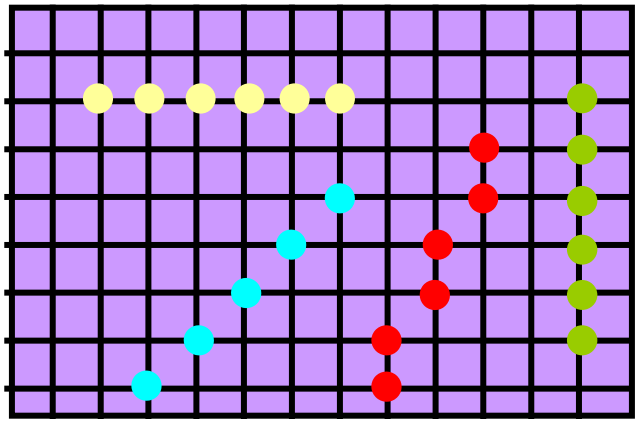
Discrete Line

- **Lines vs. Line Segments**
- **What is a discrete line segment?**
  - This is a relatively recent problem
  - How to generate a discrete line?



A Good Line

- **No gaps in adjacent pixels**
- **Pixels close to ideal line**
- **Consistent choices; same pixels in same situations**



A Good Line

- **Smooth looking**
- **Even brightness in all orientations**
- **Same line for** $P_0 P_1$ **as for** $P_1 P_0$
- **Double pixels stacked up?**

"Good" Discrete Line - 2

- **Smooth looking**
- **Even brightness in all orientations**
- **Same line for** $P_0 P_1$ **as for** $P_1 P_0$
- **Double pixels stacked up?**

Incremental Fn Eval

- **Recall** $f(x_{i+1}) = f(x_i) + \Delta(x_i)$
- **Characteristics**
  - Fast
  - Cumulative Error
- **Need to define** $f(x_o)$

Restricted Form

- **Line segment in *first* octant with**

$$0 < m < 1$$

- **Let us proceed**

Two Line Equations

- **Explicit:** $y = mx + B$
- **Implicit:** $F(x, y) = ax + by + c = 0$

Define: $dy = y_1 - y_0$

$dx = x_1 - x_0$

Hence, $y = \left(\frac{dy}{dx}\right)x + B$

Relating Explicit to Implicit Eq's

Recall,

$$\frac{dy}{dx}x - y + B = 0$$

Or,

$$(dy)x + (-dx)y + (dx)B = 0$$

$$\therefore \quad F(x,y) = (dy)x + (-dx)y + (dx)B = 0$$

where,

$$a = (dy); \quad b = -(dx); \quad c = B(dx)$$

Investigate Sign of **F**

Verify that

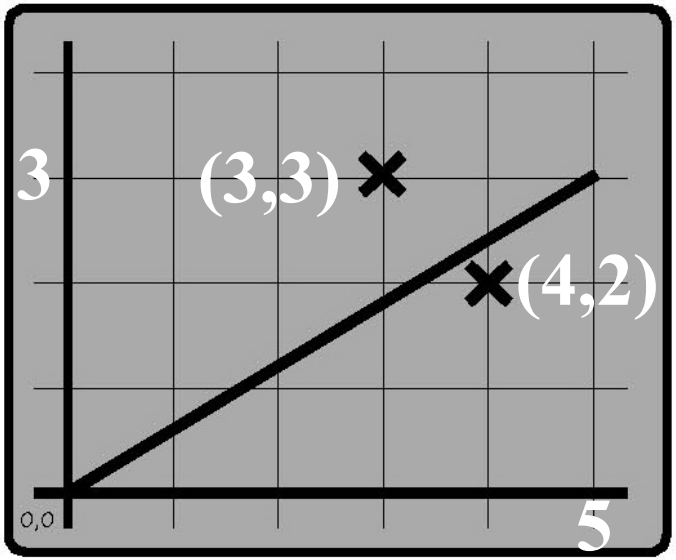
$$F(x,y) = \begin{cases} + & \text{below line} \\ 0 & \text{on line} \\ - & \text{above line} \end{cases}$$

Look at extreme values of y

Implicit Line Equation

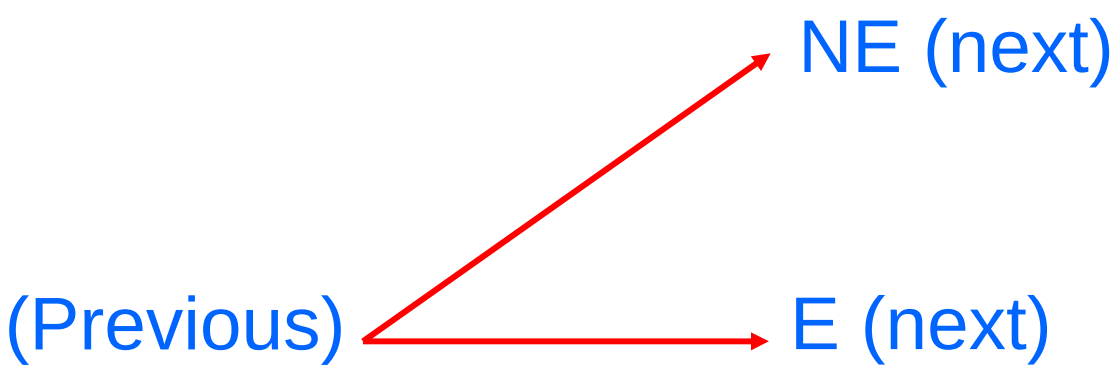
F(x,y) = ax + by + c = 0
a=dy b = -dx and c=B(dx)
dx=5 dy=3 B=0

E.g. line (0,0) to (5,3)
f(3,3) = 3*3 – 5*3 = -6 <0
f(4,2) = 3*4 – 5*2 = 2 >0



Key to Bresenham Algorithm

"Reasonable assumptions" have reduced the problem to making a binary choice at each pixel:

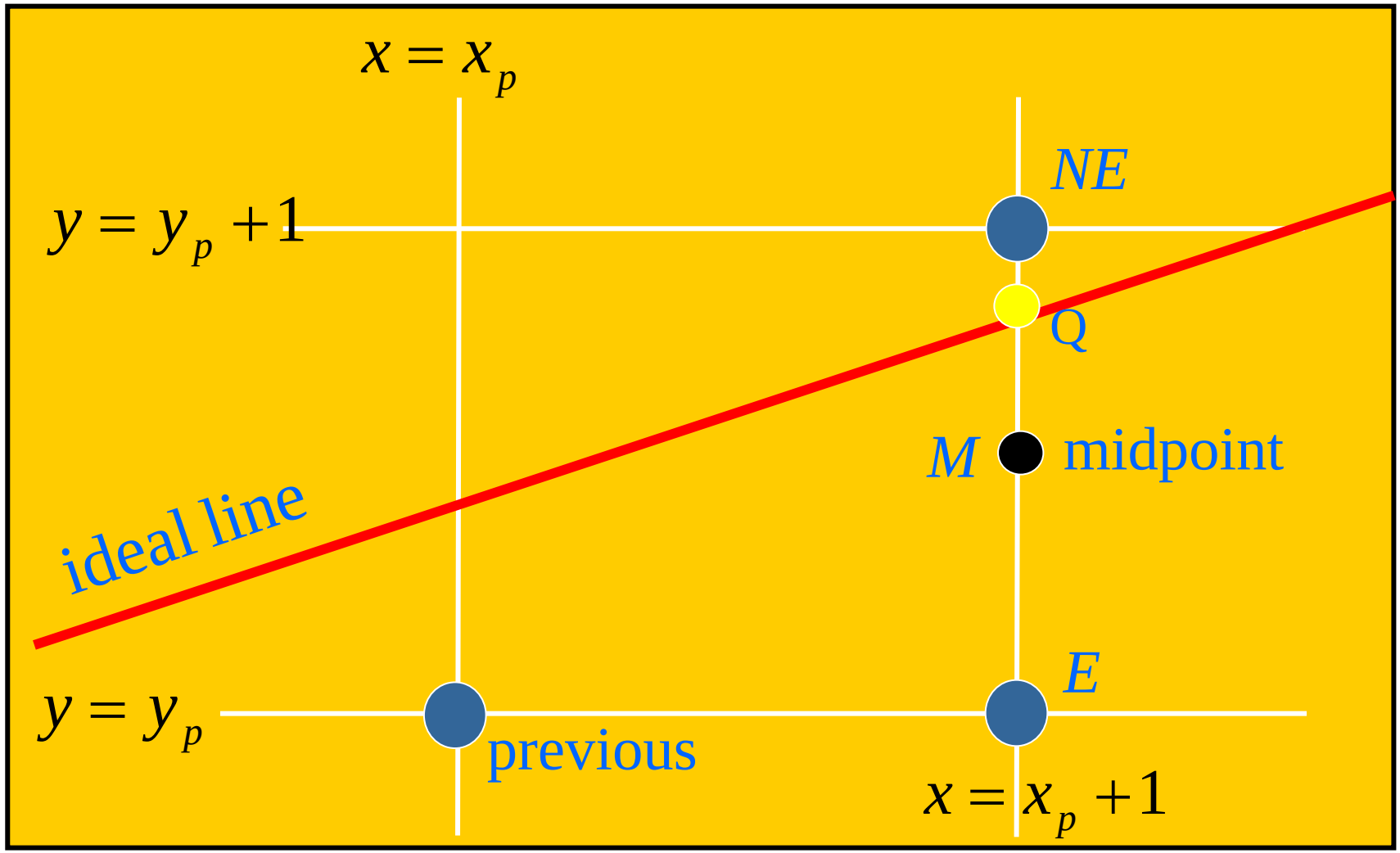


Decision Variable d *(logical)*

Define a logical *decision* variable *d*

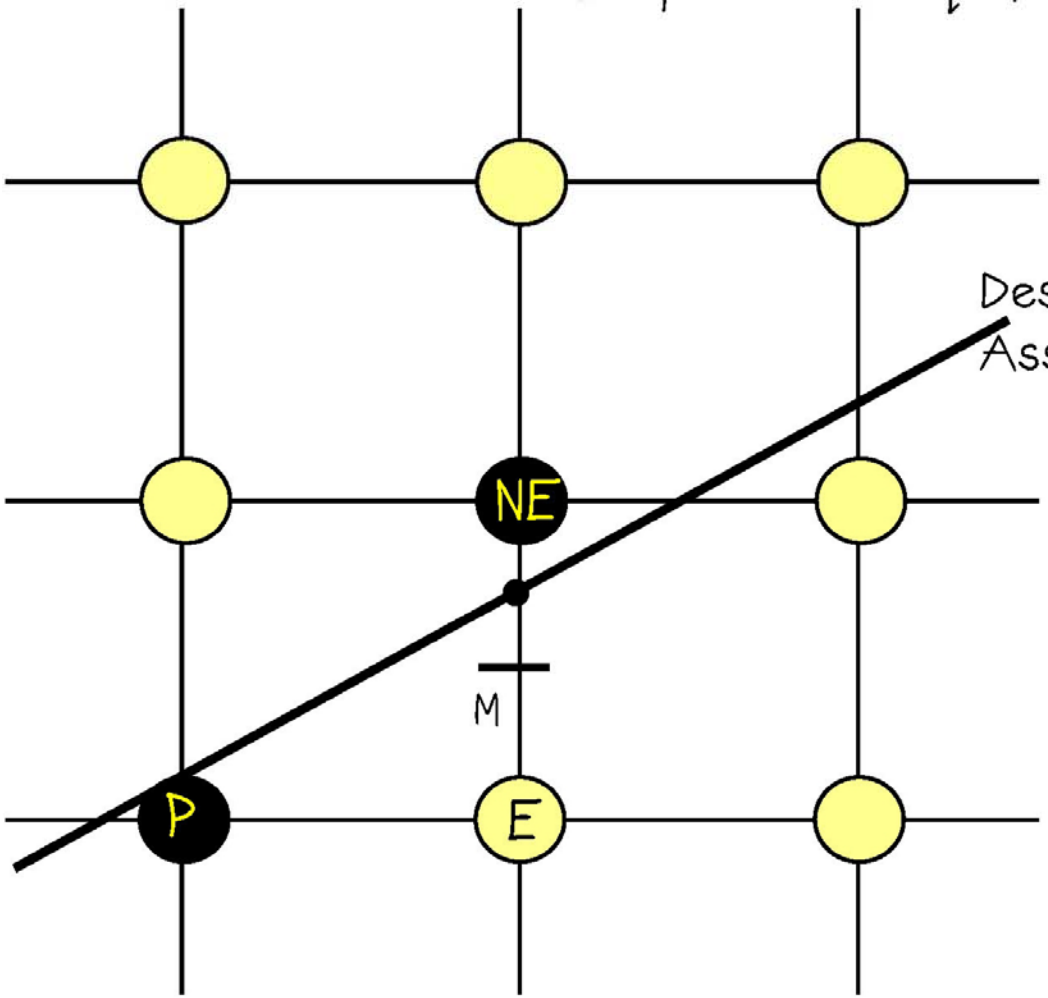
- linear in form
- incrementally updated (with addition)
- tells us whether to go *E* or *NE*

The Picture



Bresenham's Algorithm (Jack Bresenham 1965)

(midpoint technique) **Pittaway**

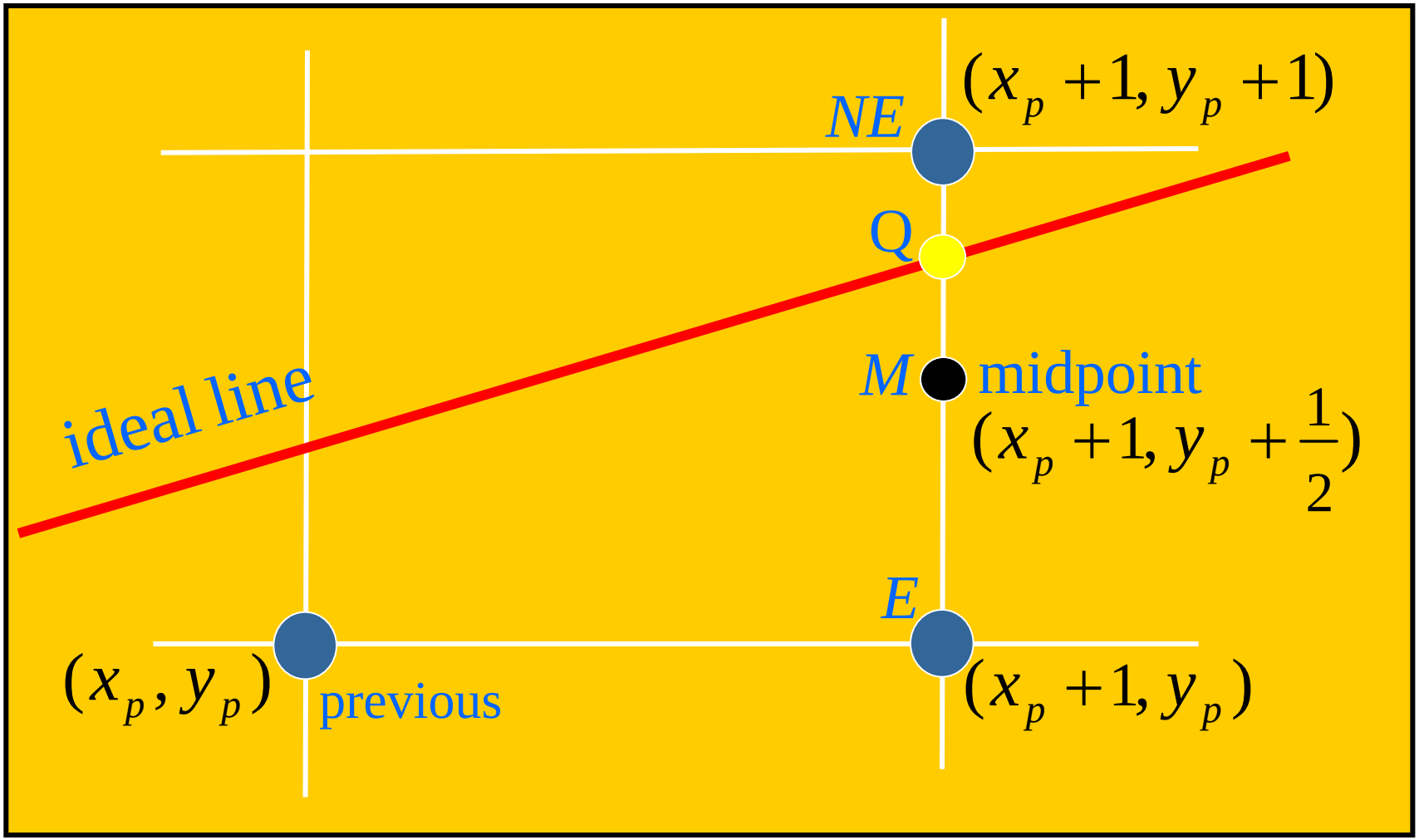


Desired Line
Assume 0 >= slope >= 1

Suppose we have chosen pixel P, which pixel should be next?

Since the slope is between 0 and 1 the next pixel will be either NE or E.

The Picture (again)



Observe the relationships

- **Suppose Q is above M, as before.**
- **Then** $F(M) > 0$, ***M* is below the line**
- **So,** $F(M) > 0$ **means line is above M,**
- **Need to move *NE, increase y* value**

Observe the relationships

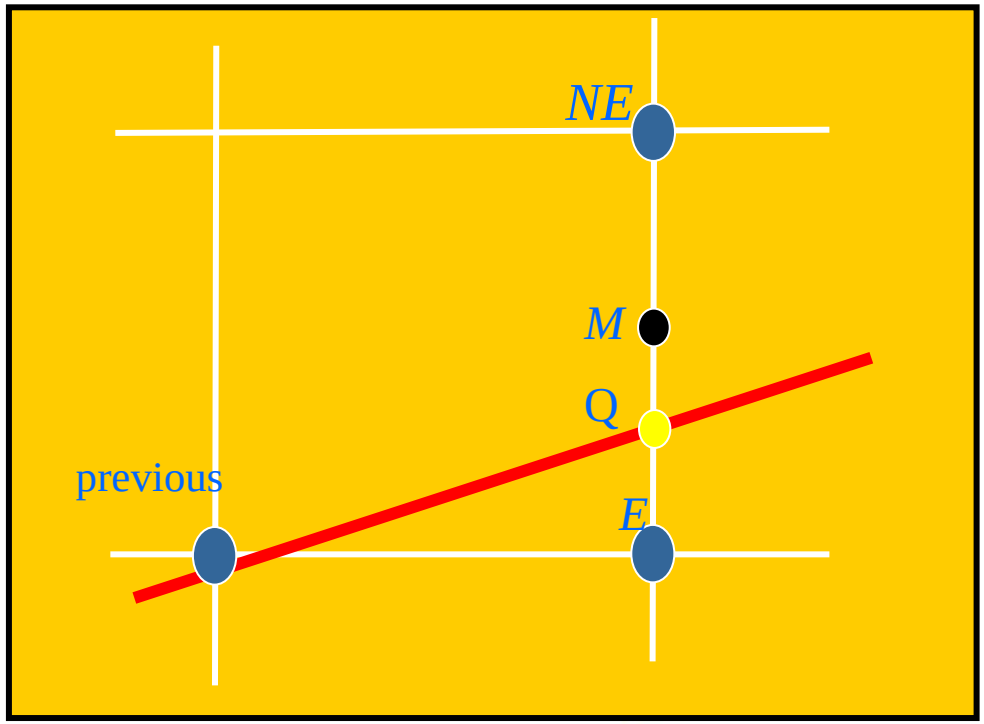
- **Suppose Q is below M.**
- **Then $F(M) < 0$, implies M is *above* the line**

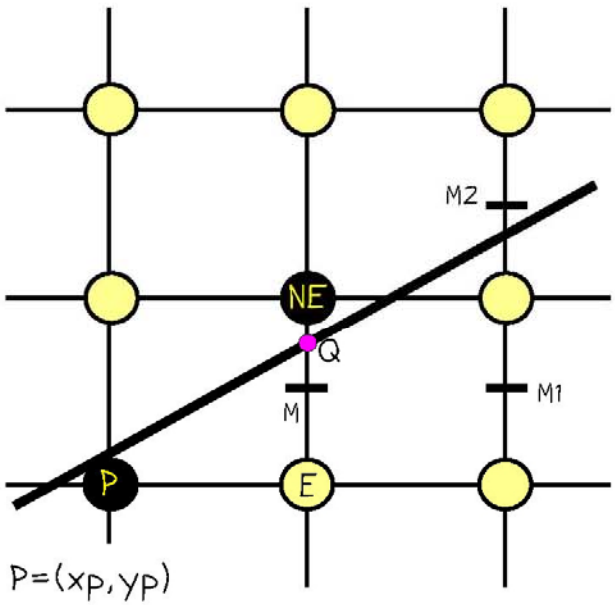
- **So, $F(M) < 0$,**

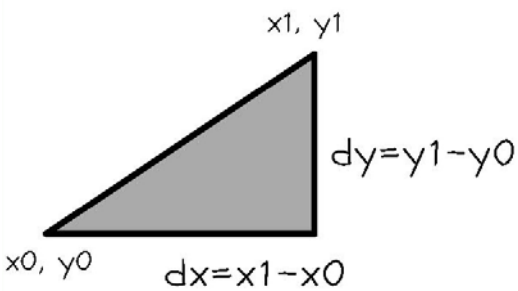
means line is below M,

- **Need to move to *E;***

don't increase y







How to choose B or C, is Q closer to NE or E?

Slope = dy/dx

We need to compute $F(M) = F(xp+1,\ yp+\frac{1}{2})$

Define <u>discriminant</u> $d = F(xp+1,\ yp+\frac{1}{2})$

$d_M = a.(xp+1) + b.(yp+\frac{1}{2}) + c$

We wish to make an incremental algorithm:

if d > 0 choose pixel NE, find M2

new $d_{M2} = F(xp+2,\ yp+\frac{3}{2}) = a.(xp+2) + b.(yp+\frac{3}{2}) + c$

if d < 0 choose pixel E, find M1

new $d_{M1} = F(xp+2,\ yp+\frac{1}{2}) = a.(xp+2) + b.(yp+\frac{1}{2}) + c$

Calculating the disciminant d

if $d < 0$ choose pixel E,

new $d_{M1} = a.(xp+2) + b.(yp+\frac{1}{2}) + c$

old $d_M = a.(xp+1) + b.(yp+\frac{1}{2}) + c$

Subtracting: $d_{M1} = d_M + a$
but $a = dy$ known as ΔE

if $d > 0$ choose pixel NE, find M2

new $d_{M2} = F(xp+2, yp+\frac{3}{2})$
$= a.(xp+2) + b.(yp+\frac{3}{2}) + c$

Subtracting d_M gives:

$d_{M2} = d_M + a + b$
$\Delta NE = a + b = dy - dx$

Initial value of d

First pixel is endpoint xo, yo
First midpoint is $(xo+1, yo+\frac{1}{2})$
Find d by choosing E or NE:

$F(xo+1, yo+\frac{1}{2}) = a(xo+1) + b(yo+\frac{1}{2}) + c$
$= axo + byo + c + a + b/2$
$= F(xo, yo) + a + b/2$

Since F(xo, yo) is on the line F(xo, yo) = 0
initial value of $d = a + b/2 = dy - dx/2$

Since we don't want the $\frac{1}{2}$ in the expression multiply by 2 which has no effect on the sign of d:

initial $d = 2dy - dx$

Increment of $\Delta E = 2dy$
Increment of $\Delta NE = 2(dy - dx)$

```
void midpoint-line(int xo, int yo, int xe,
                   int ye, int value)
{
    int dx = xe - xo;
    int dy = ye - yo;
    int d = 2 * dy - dx;
    int incrE = 2 * dy;
    int icrNE = 2 * (dy - dx);
    int x = xo;
    int y = yo;
    writePixel(xo,yo, value);
```

```
while (x<xe) {
    if (d<=0) {   /* choose E */
        d+=incrE;
        x++;
    } else {      /* choose NE */
        d += incrNE;
        x++;
        y++;
    }
    writePixel(x,y, value);
} /* end while */

} /* end midPoint line */
```

Example

- **Line end points:**

$$(x_0, y_0) = (5,8); \quad (x_1, y_1) = (9,11)$$

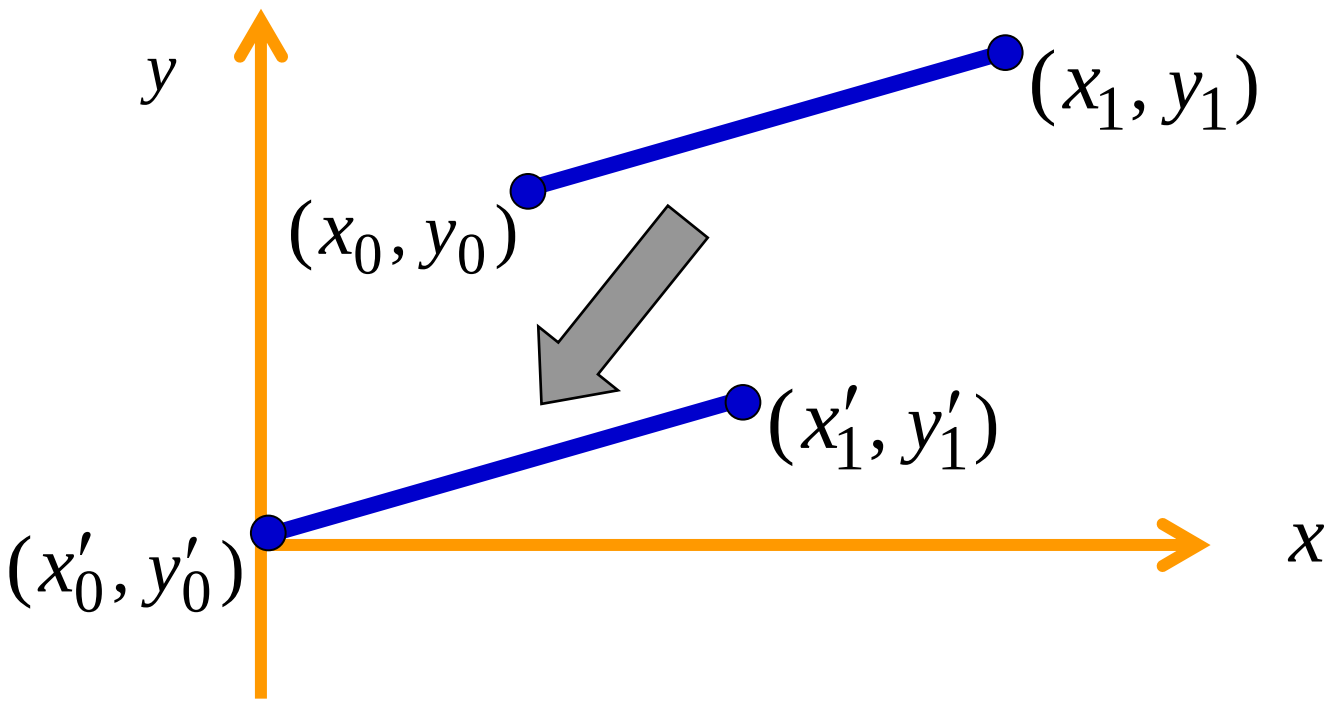
- **Deltas:**

$$dx = 4; dy = 3$$

Meeting Bresenham Criteria

- $m = 0; \quad m = 1 \quad \Rightarrow$ trivial cases
- $(x_0, y_0) \neq (0,0) \quad \Rightarrow$ translate
- $0 > m > -1 \quad \Rightarrow$ flip about x-axis
- $m > 1 \quad \Rightarrow$ flip about $x = y$

Case 1: Translate to Origin



Case 0: Trivial Situations

- $m = 0 \quad \Rightarrow \quad \text{horizontal line}$
- $m = 1 \quad \Rightarrow \quad \text{line } \; y = x$

Do not need Bresenham

Case 1: Translate to Origin

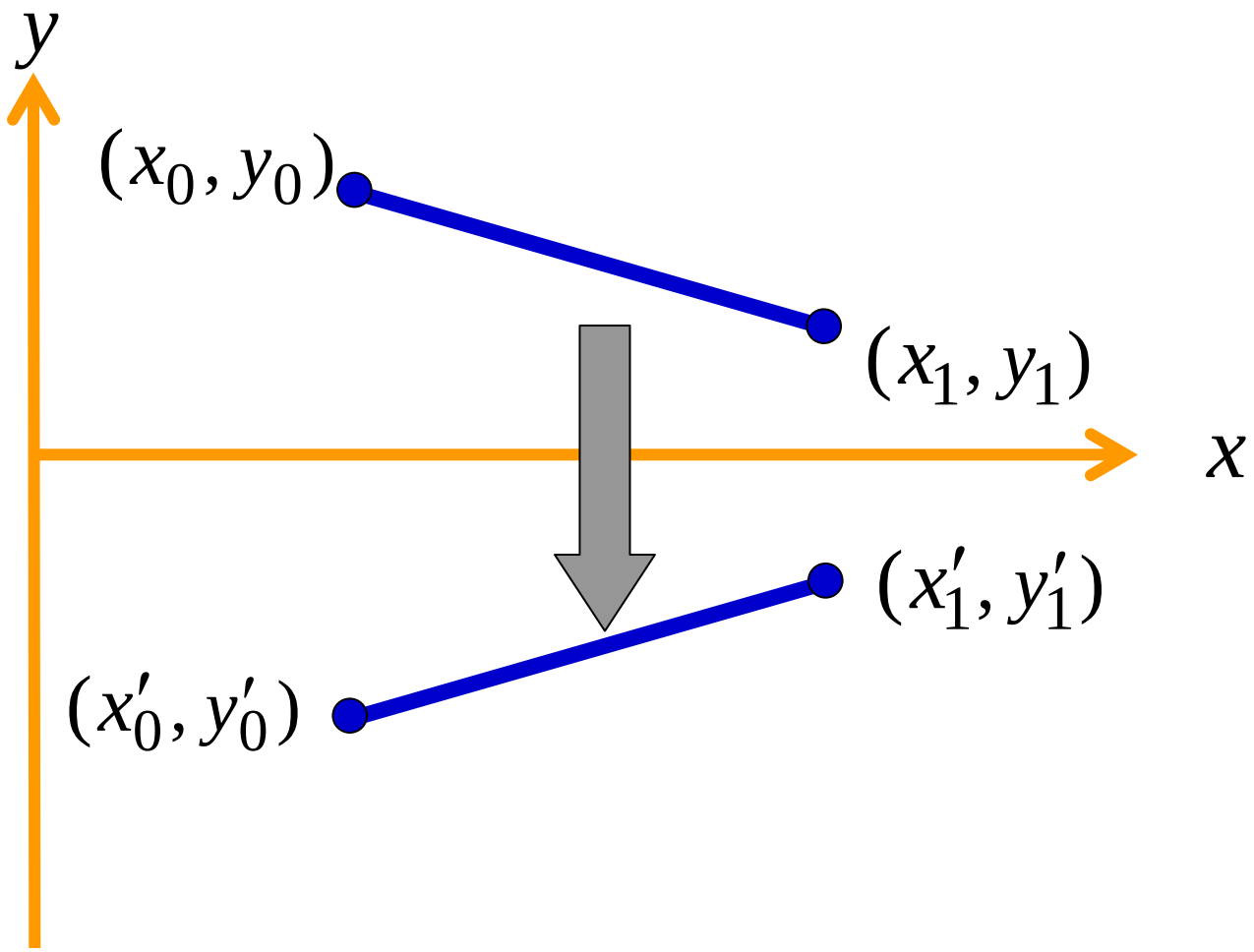
- Move (x_0, y_0) to the origin

$$(x_0', y_0') = (0,0);$$

$$(x_1', y_1') = (x_1 - x_0, y_1 - y_0)$$

- **Need only consider lines emanating from the origin.**

Case 2: m > -1 Flip about x-axis



Case 2: Flip about x-axis

- Suppose, $0 > m > -1$,
- Flip about x-axis $(y' = -y)$:

$$(x_0', y_0') = (x_0, -y_0);$$

$$(x_1', y_1') = (x_1, -y_1)$$

How do slopes relate?

$$\left.\begin{aligned} m &= \frac{y_1 - y_0}{x_1 - x_0}\,; \\ m' &= \frac{y'_1 - y'_0}{x_1 - x_0} \end{aligned}\right\} \quad \text{by definition}$$

Since $y'_{\mathrm{i}} = -y_{\mathrm{i}}$, $\quad m' = \dfrac{-y_1 - (-y_0)}{x_1 - x_0}$

How do slopes relate?

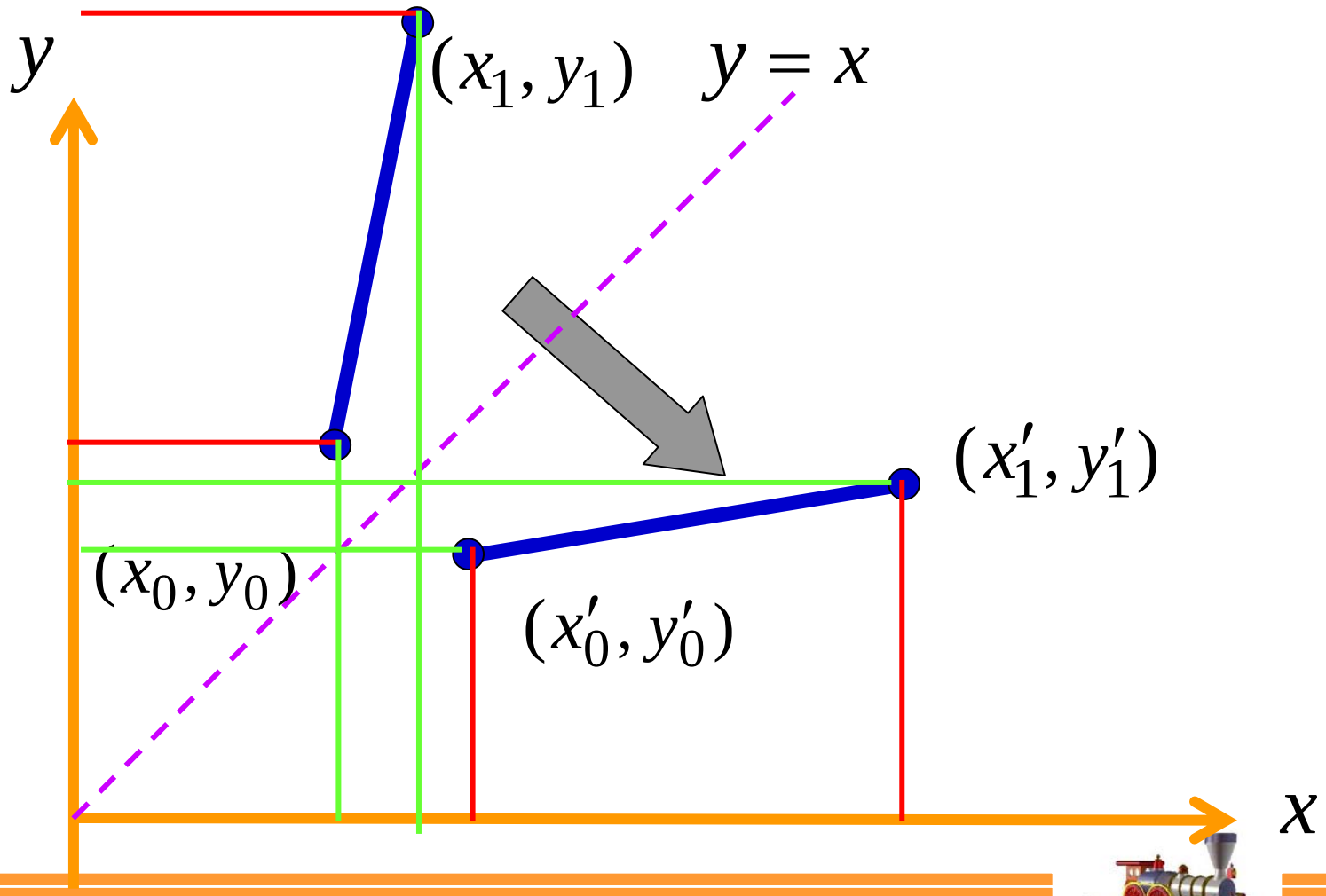
$$m' = -\frac{(y_1 - y_0)}{x_1 - x_0}$$

i.e.,

$$m' = -m$$

$$\boxed{\therefore \quad 0 > m > -1 \quad \Rightarrow \quad 0 < m' < 1}$$

Case 3: m > 1: Flip about line y = x



Case 3: Flip about line y = x

$$y = mx + B,$$

$$\text{swap} \quad x \leftrightarrow y \quad \text{and} \quad \text{prime} \quad \text{them} \quad ,$$

$$x' = my' + B,$$

$$my' = x' - B$$

Case 3: m′ = ?

$$y' = \left(\frac{1}{m}\right) x' - B,$$

$$\therefore \quad m' = \left(\frac{1}{m}\right) \quad \text{and,}$$

$$\boxed{m > 1 \quad \Rightarrow \quad 0 < m' < 1}$$

Example

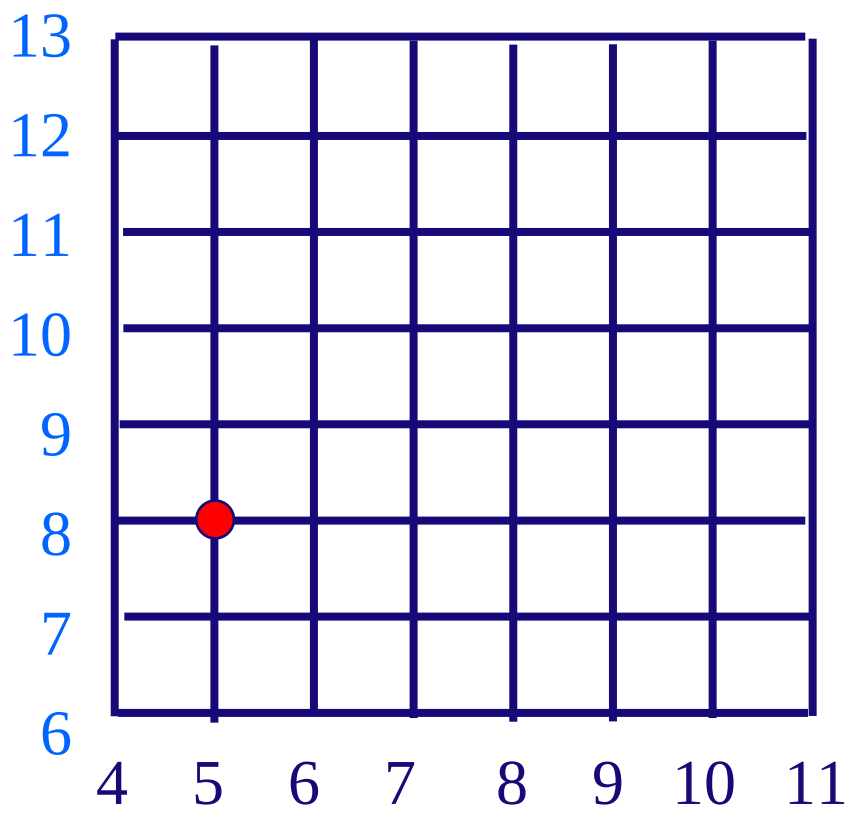
- **Line end points:**

$$(x_0, y_0) = (5,8); \quad (x_1, y_1) = (9,11)$$

- **Deltas:** $$dx = 4; dy = 3$$

- **After translation: (0,0) → (4,3)**

Graph



Example (dx = 4; dy = 3)

- **Initial value of** $dx = 4; dy = 3$

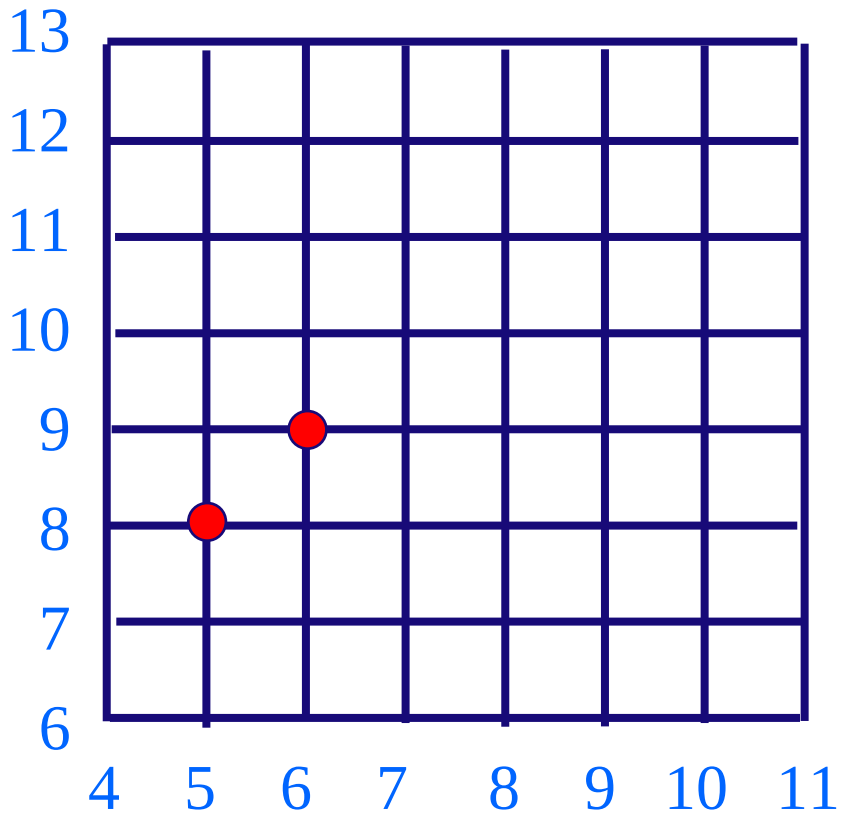
d = 2dy – dx
d = 2*3 – 4 = 2 > 0

incrE = 2*dy = 6
incrNE = 2*(-1) = -2

so NE is first move

```
void midpoint-line(int xo, int yo, int xe,
                   int ye, int value)
{
    int dx = xe - xo;
    int dy = ye - yo;
    int d = 2 * dy - dx;
    int incrE = 2 * dy;
    int icrNE = 2 * (dy - dx);
    int x = xo;
    int y = yo;
    writePixel(xo,yo, value);
```

Graph



Example (dx = *4*; dy = *3*)

- **Last move was NE**

Update value of *d*

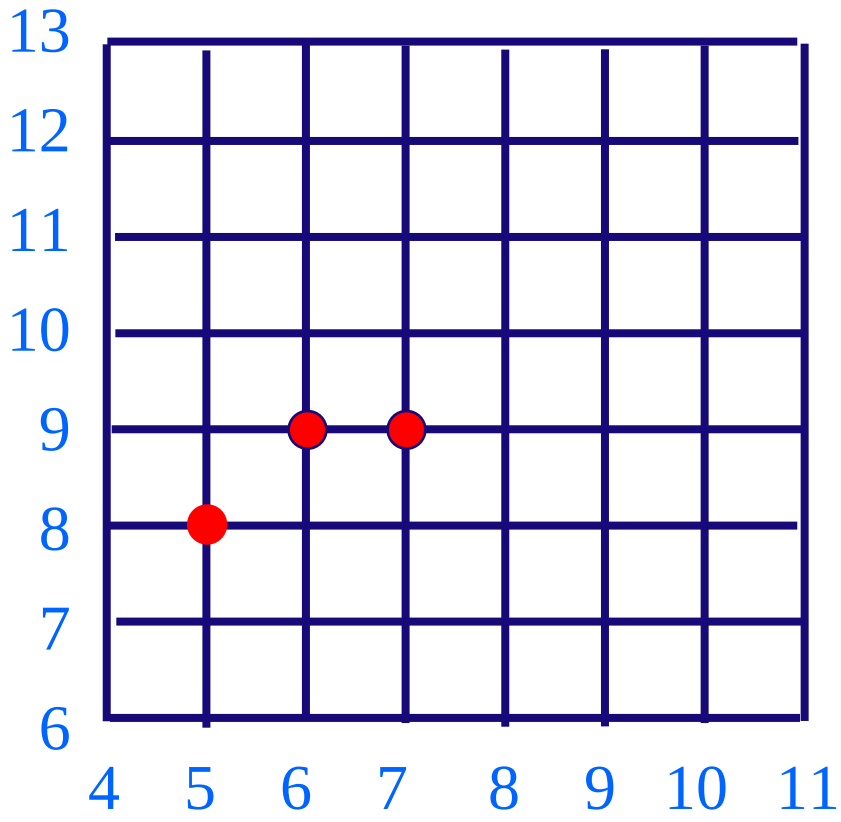
- **d = 2 + incrNE = 2 – 2 = 0**

So move East

```
while (x<xe) {
        if (d<=0) {   /* choose E */
            d+=incrE;
            x++;
        } else {      /* choose NE */
            d += incrNE;
            x++;
            y++;
        }
        writePixel(x,y, value);
} /* end while */

} /* end midPoint line */
```

Graph



Example (dx = *4*; dy = *3)*

- **Last move was E**

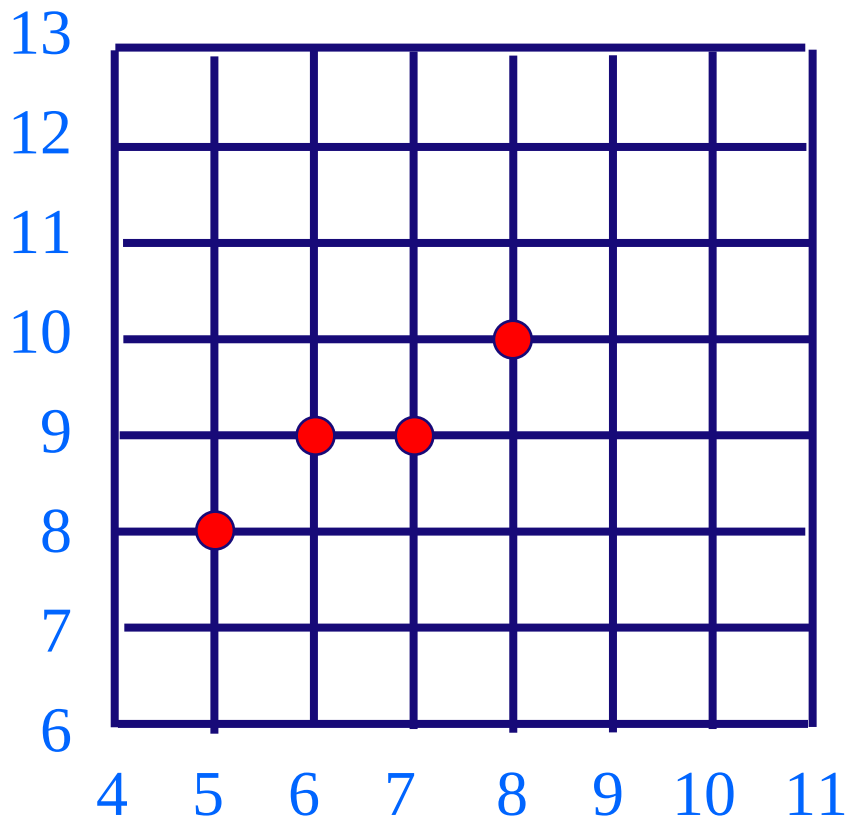
Update value of *d*

- **d = 0 + incrE = 6**

So move NE

```
while (x<xe) {
      if (d<=0) {   /* choose E */
          d+=incrE;
          x++;
      } else {      /* choose NE */
          d += incrNE;
          x++;
          y++;
      }
      writePixel(x,y, value);
} /* end while */

} /* end midPoint line */
```


13
12
11
10
9
8
7
6
4 5 6 7 8 9 10 11

Example (dx = *4*; dy = *3*)

- **Last move was NE**

Update value of *d*

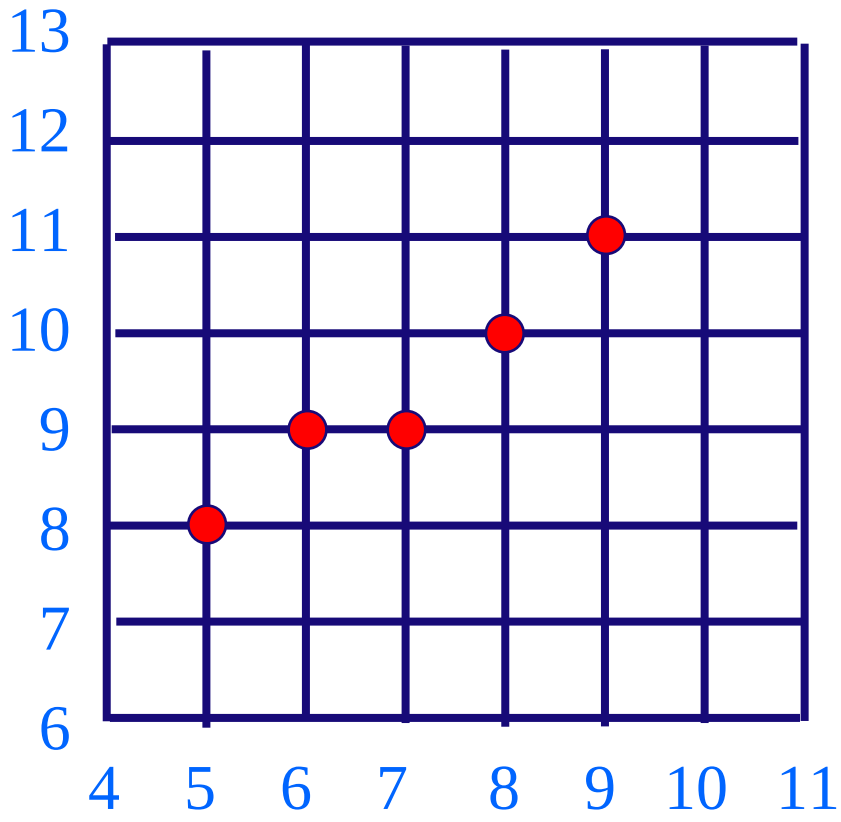
- **d = 6 + incrNE = 4**

So move NE

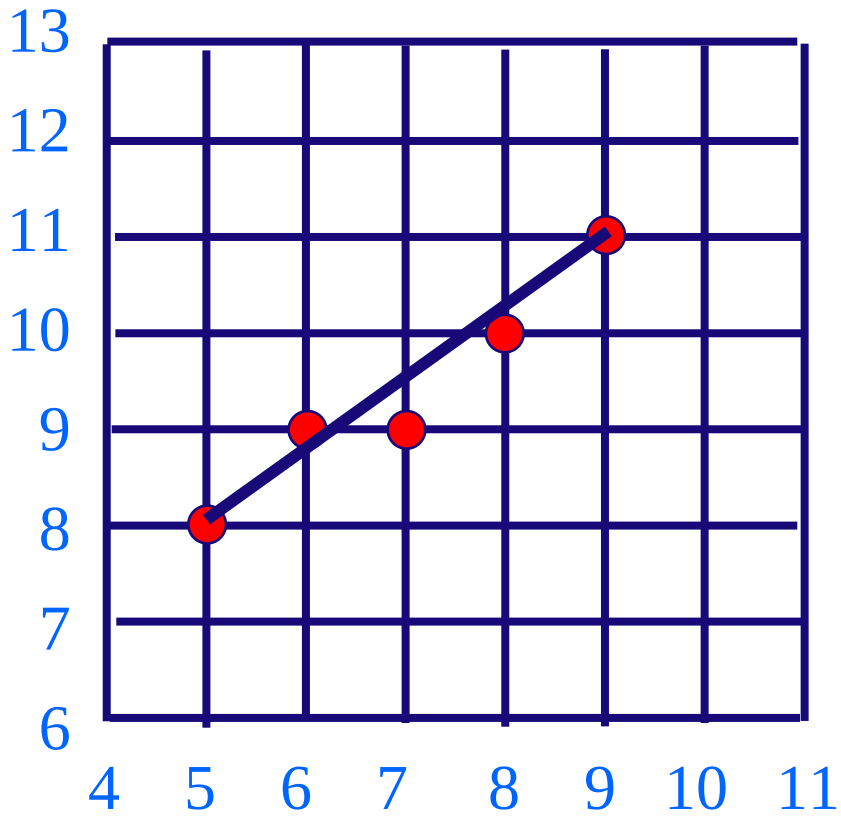
```
while (x<xe) {
      if (d<=0) {   /* choose E */
          d+=incrE;
          x++;
      } else {      /* choose NE */
          d += incrNE;
          x++;
          y++;
      }
      writePixel(x,y, value);
} /* end while */

} /* end midPoint line */
```

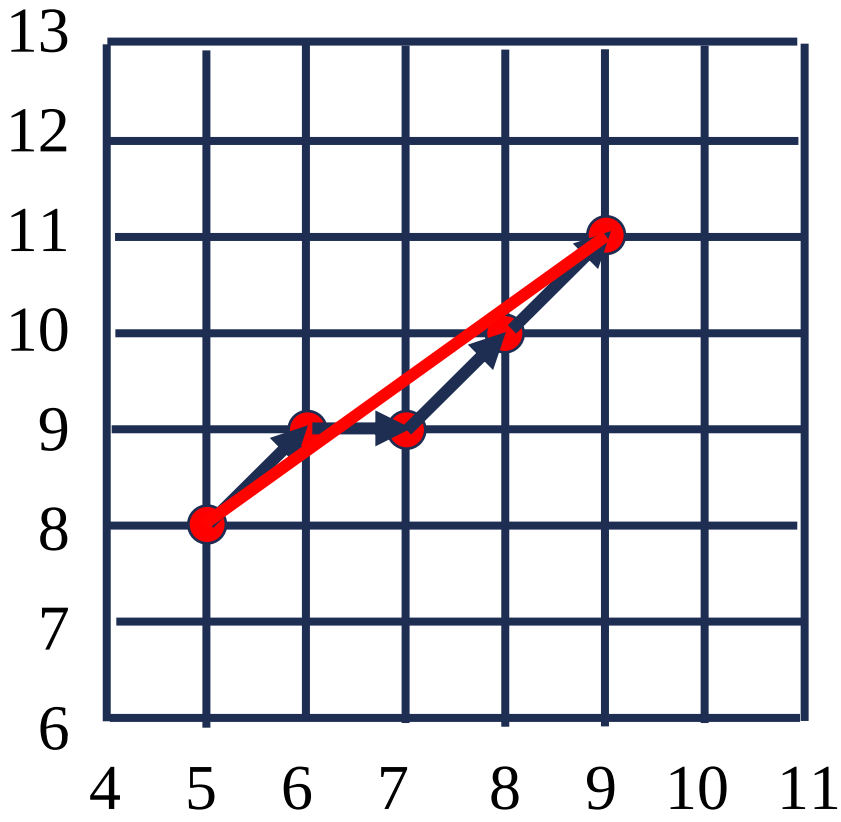
Graph



Graph



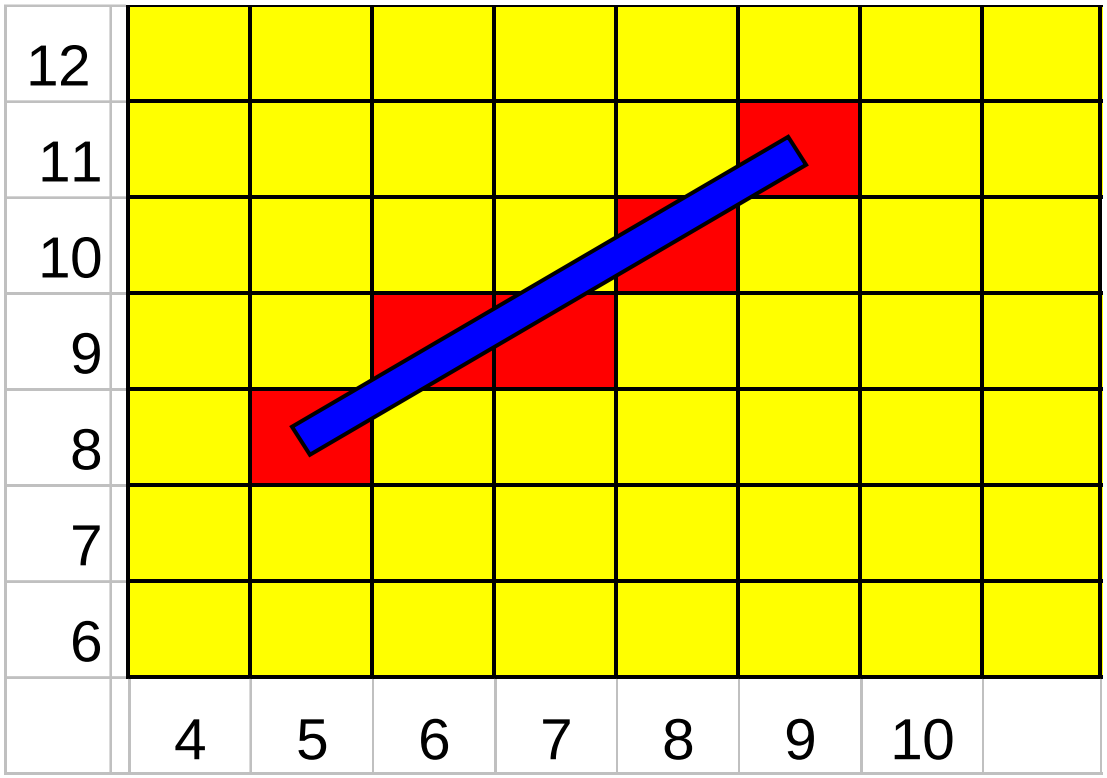
Graph



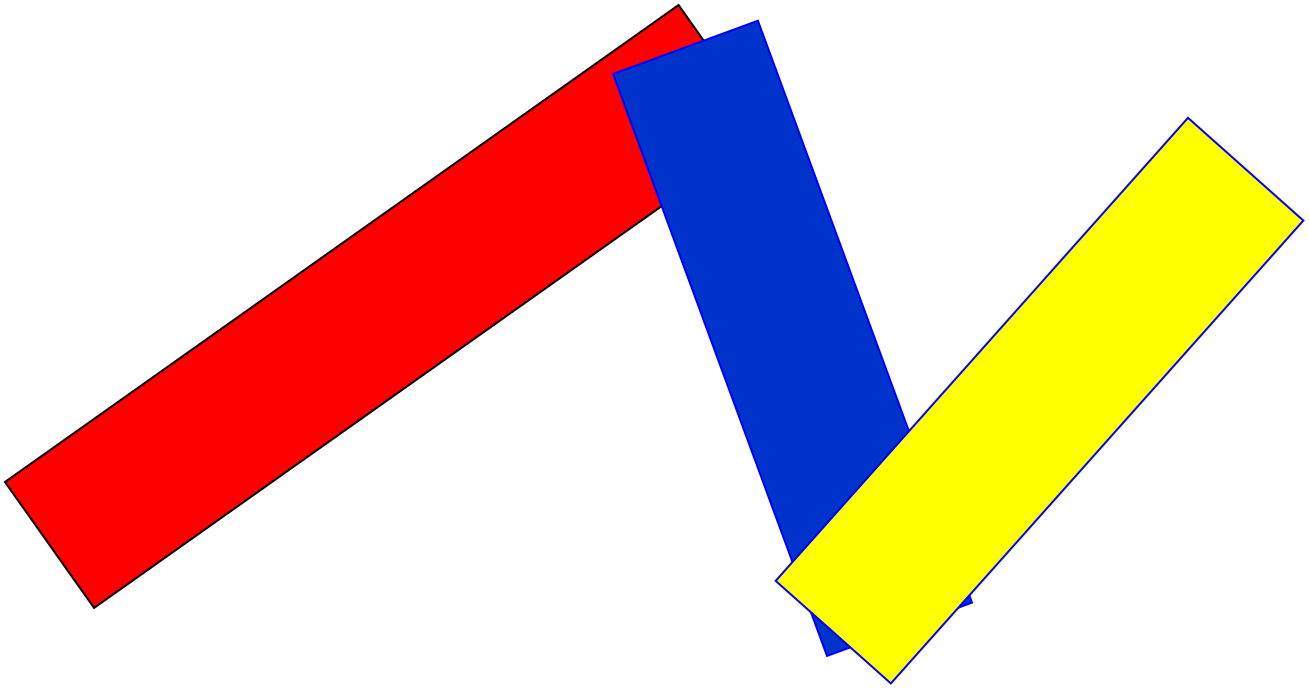
More Raster Line Issues

- **Fat lines with multiple pixel width**
- **Symmetric lines**
- **How should end pt geometry look?**
- **Generating curves, e.g., circles, etc.**
- **Jaggies, staircase effect, aliasing...**

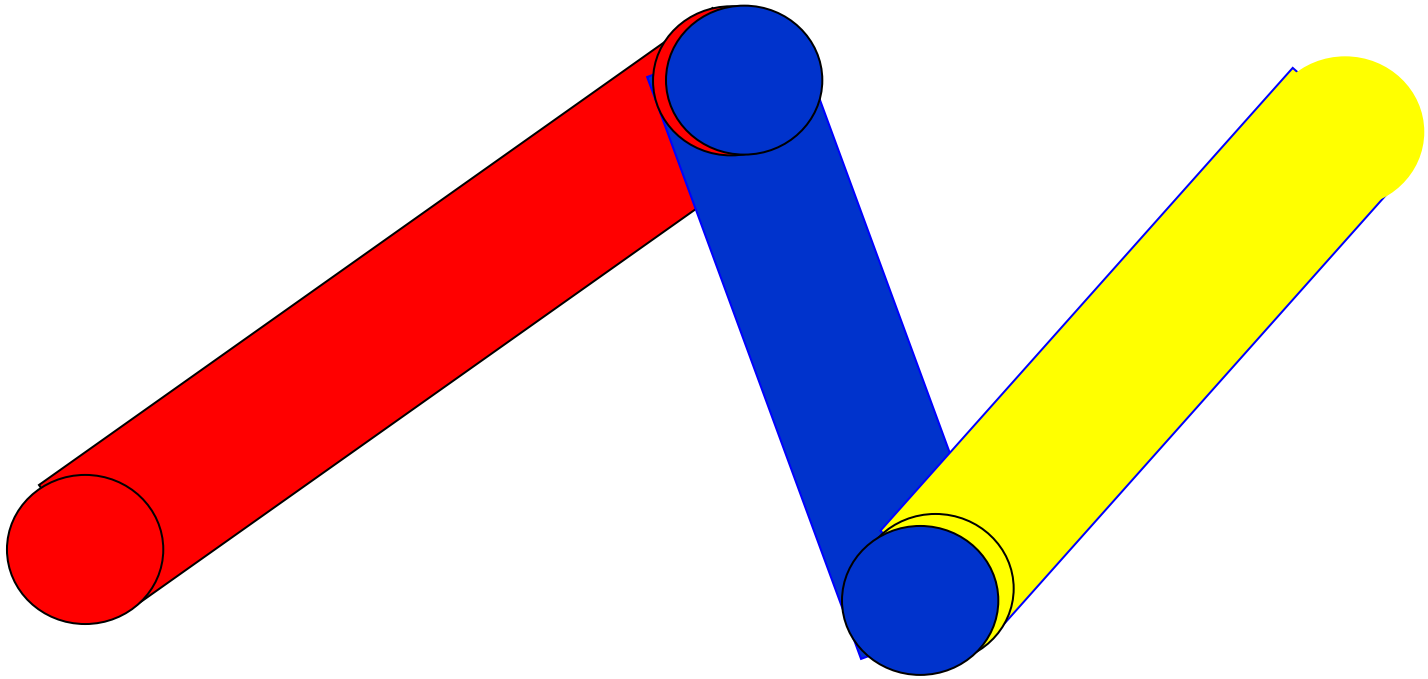
Pixel Space



Example

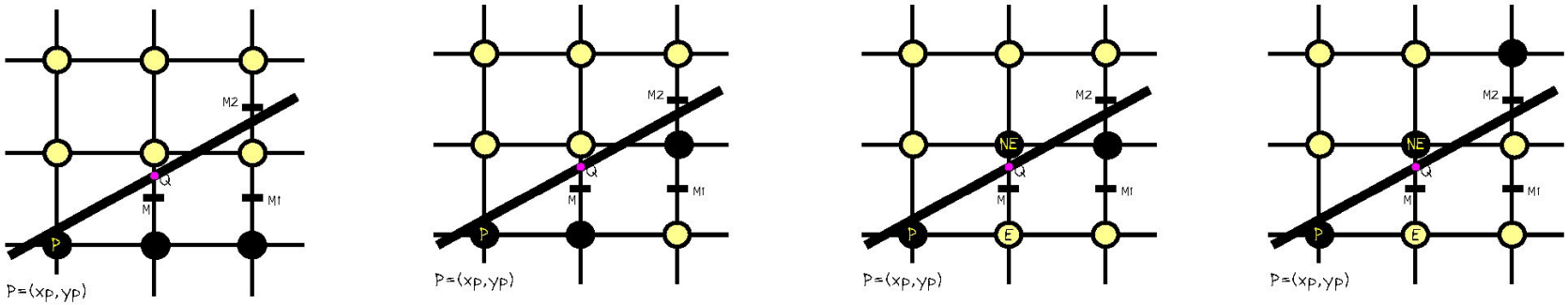


Example



Double Step Line Algorithm X Wu (1987)

Symmetric Variation (blob 1987)



Pattern 1 Pattern 2 Pattern 3 Pattern 4

Bresenham's chooses next pixel with one discriminator.
Wu chooses next pattern of two pixels with one discriminator.
By drawing from both ends of line at once 4 pixels can be chosen with one discriminator.

Observe that for 0 < slope <= 1/2 choose pattern 1 or (pattern 2 or 3)
for 1/2 < slope <= 1 choose pattern 4 or (pattern 2 or 3)

This can be done with only one discriminator calculated in the loop.
See code in course directory. For a full derivation see:
Fast Line Scan-Conversion, J.G. Rokne and B. Wyvill and X. Wu,
"ACM Transactions on Graphics", pp376-388, 9, 4, 1990

Double Step code for slope < 1/2

```
for (i = 0; i < xend; i++) {     /* plotting loop */
        ++x;
        --x1;
        if (D < 0) {  /* pattern 1 forwards */
                plot(x, y, reverse);
                plot(++x, y, reverse);
                  /* pattern 1 backwards */
                plot(x1, y1, reverse);
                plot(--x1, y1, reverse);
                D += incr1;
        } else {
                if (D < c) { /* pattern 2 forwards */
                        plot(x, y, reverse);
                        plot(++x, y += step, reverse);
                        /* pattern 2 backwards */
                        plot(x1, y1, reverse);
                        plot(--x1, y1 -= step, reverse);
                }  else {
                        /* pattern 3 forwards */
                        plot(x, y += step, reverse);
                        plot(++x, y, reverse);
                        /* pattern 3 backwards */
                        plot(x1, y1 -= step, reverse);
                        plot(--x1, y1, reverse);
                }
                D + = incr2; /* only one discriminator incremented */
        }
}/* end for */
```