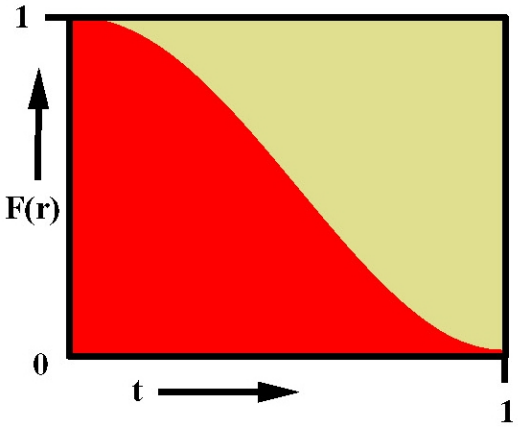
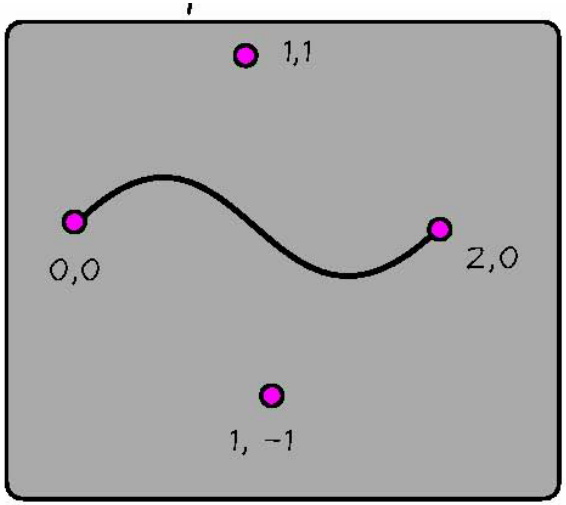
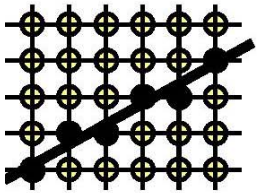
Computer Graphics

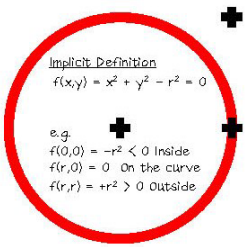


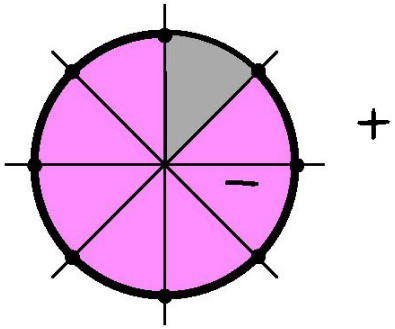




Scan Conversion

Polynomials





by
Brian Wyvill

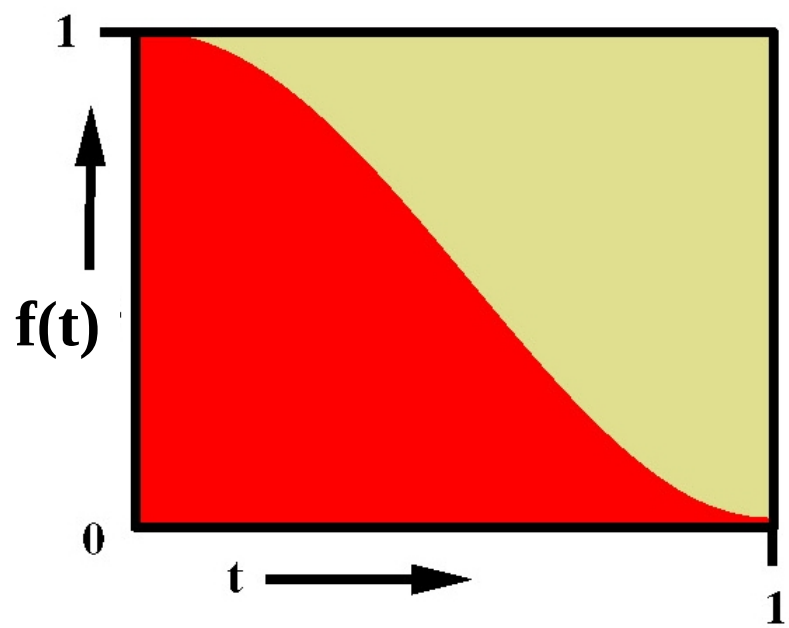
Parametric Curve

$$f(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$ **similarly for y and z**

Straightforward implementation
by Horner's Rule
e.g.

```
void x(double t)
{
return t*(t*(t*a_x+b_x)+c_x)+d_x;
}
```

5 multiplies 3 additions



Scan Conversion Of Polynomials

$$f(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

<u>Repeated Evaluation of Cubic by Forward Differences</u>

Definition : $\Delta f(t) = f(t+\delta) - f(t) \quad \delta > 0$

rewriting: $f(t+\delta) = \Delta f(t) + f(t)$

Rewriting iteratively: $f_{n+1} = f_n + \Delta f_n$

Definition:

$$\boxed{\Delta f(t)=f(t+\delta)-f(t)}$$

$$f(t) = at^3+bt^2+ct+d$$

$$\Delta f(t) = a(t+\delta)^3 + b(t+\delta)^2 + c(t+\delta) + d - (at^3+bt^2+ct+d)$$

$$\boxed{\Delta f(t) = 3at^2\delta+t(3a\delta^2+2b\delta)+a\delta^3+b\delta^2+c\delta \quad --(1)}$$

So $\Delta f(t)$ is second degree. Applying forward differences again to reduce this further:

$$\Delta^2 f(t) = \Delta(\Delta f(t)) = \Delta f(t+\delta)-\Delta f(t)$$

applying by writing $(t+\delta)$ for t in (1)
$\Delta f(t)$ or $(3at^2\delta+t(3a\delta^2+2b\delta)+a\delta^3+b\delta^2+c\delta)$

$$\boxed{\text{Yields } \Delta^2 f(t) = 6a\delta^2 t + 6a\delta^3 + 2b\delta^2 \quad --(2)}$$

$$\Delta^3 f(t) = \Delta(\Delta^2 f(t)) = \Delta^2 f(t+\delta)-\Delta^2 f(t)$$

substituting $(t+\delta)$ for t in (2)

$$\boxed{\Delta^3 f(t) = 6a\delta^3 \quad --(3)}$$

By Definition : $\Delta f(t)=f(t+\delta)-f(t)$

rewriting: $f(t+\delta)=f(t)+\Delta f(t)$

$$\boxed{f_{n+1} = f_n + \Delta f_n}$$

In other words $t_n = n\delta$ and $f_n=f(t_n)$

f is evaluated at equal intervals of size δ.

At t=0

$$\begin{aligned} f_0 &= d \\ \Delta f_0 &= a\delta^3+b\delta^2+c\delta \\ \Delta^2 f_0 &= 6a\delta^3 + 2b\delta^2 \\ \Delta^3 f_0 &= 6a\delta^3 \end{aligned}$$

by definition: $\Delta^2 f(t) = \Delta f(t+\delta)-\Delta f(t)$

or $\Delta^2 f_n = \Delta f_{n+1} - \Delta f_n$

$$\boxed{\Delta f_{n+1} = \Delta f_n + \Delta^2 f_n}$$

similarly

$$\Delta^3 f(t) = \Delta^2 f(t+\delta)-\Delta^2 f(t)$$

$$\boxed{\Delta^2 f_{n+1} = \Delta^2 f_n + \Delta^3 f_n}$$

and $\Delta^3 f_n$ is constant

Repeat following steps m times with n initially 0:-

$$\begin{aligned} f_{n+1} &= f_n + \Delta f_n \\ \Delta f_{n+1} &= \Delta f_n + \Delta^2 f_n \\ \Delta^2 f_{n+1} &= \Delta^2 f_n + \Delta^3 f_0 \end{aligned}$$

n=0

$$\begin{aligned} f_1 &= f_0 + \Delta f_0 \\ \Delta f_1 &= \Delta f_0 + \Delta^2 f_0 \\ \Delta^2 f_1 &= \Delta^2 f_0 + \Delta^3 f_0 \end{aligned}$$

n=1

$$\begin{aligned} f_2 &= f_1 + \Delta f_1 \\ \Delta f_2 &= \Delta f_1 + \Delta^2 f_1 \\ \Delta^2 f_2 &= \Delta^2 f_1 + \Delta^3 f_0 \end{aligned}$$

Suppose we want 64 steps

n=64 or δ=1/64

$t_0 = 0\delta = 0$

$t_1 = 1\delta = 1/64$

$t_2 = 2\delta = 2/64$ etc.

Forward Differences

continued

$$f_0 = d$$
$$\Delta f_0 = a\delta^3+b\delta^2+c\delta$$
$$\Delta^2 f_0 = 6a\delta^3 + 2b\delta^2$$
$$\Delta^3 f_0 = 6a\delta^3$$

$$\begin{bmatrix} f_0 \\ \Delta f_0 \\ \Delta^2 f_0 \\ \Delta^3 f_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \delta^3 & \delta^2 & \delta & 0 \\ 6\delta^3 & 2\delta^2 & 0 & 0 \\ 6\delta^3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$D_x = E(\delta)C_x \qquad \begin{matrix} f_0 \\ \Delta f_0 \\ \Delta^2 f_0 \\ \Delta^3 f_0 \end{matrix}$$
$$D_y = E(\delta)C_y$$
$$D_z = E(\delta)C_z$$

$f_x(t) = a_x t^3+b_x t^2+c_x t+d_x$
similarly for y and z

so we have:

$$\underbrace{\begin{bmatrix} f_0 \\ \Delta f_0 \\ \Delta^2 f_0 \\ \Delta^3 f_0 \end{bmatrix}}_{\text{x and y and z}} \text{ and can calculate } \underbrace{\begin{bmatrix} f_n \\ \Delta f_n \\ \Delta^2 f_n \\ \Delta^3 f_n \end{bmatrix}}_{\text{x and y and z}}$$

algorithm writing Δx for Δf_{nx} etc.

```
for (i=0; i<n; i++) {
    x+=Δx;    Δx+=Δ²x;    Δ²x+=Δ³x;
    y+=Δy;    Δy+=Δ²y;    Δ²y+=Δ³y;
    z+=Δz;    Δz+=Δ²z;    Δ²z+=Δ³z;
    lineAbs(x,y,z);
}
```

An Example:

$$f(t) = 2t^3 - 3t^2 + 3t$$

$a = 2$
$b = -3$
$c = 3$

taking 100 steps $\delta = 0.01$

At $t=0$

$$f_0 = d = 0$$

$$\Delta f_0 = a\delta^3 + b\delta^2 + c\delta = 0.029702$$

$$\Delta^2 f_0 = 6a\delta^3 + 2b\delta^2 = -0.000588$$

$$\Delta^3 f_0 = 6a\delta^3 = 0.000012$$

Example:

$f \quad = f_0 = d = 0$

$df \quad = \Delta f_0 = a\delta^3 + b\delta^2 + c\delta = 0.029702$

$d2f = \Delta^2 f_0 = 6a\delta^3 + 2b\delta^2 = -0.000588$

$d3f = \Delta^3 f_0 = 6a\delta^3 = 0.000012$

```
for (i=0; i<n; i++) {
    f += df;      // points on the function
   df += d2f;  // next value for 1st difference
   d2f+=d3f;  // next value for 2nd difference
}
```

f	df	d2f	d3f
0.000000	0.029702	-0.000588	0.000012
0.029702	0.029114	-0.000576	0.000012
0.058816	0.028538	-0.000564	0.000012
0.087354	0.027974	-0.000552	0.000012
0.115328	0.027422	-0.000540	0.000012
0.142750	0.026882	-0.000528	0.000012 etc.