CPSC 453 Linear Algebra Review

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1 Vector Operations

1.1 Addition

Given $\vec{v}, \vec{w} \in \mathbb{R}^n$:

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

1.2 Scalar Multiplication

Given $\vec{v} \in \mathbb{R}^n, s \in \mathbb{R}$:

$$s\vec{v} = \begin{bmatrix} sv_1 \\ sv_2 \\ \vdots \\ sv_n \end{bmatrix}$$

1.3 Dot Product

Given $\vec{v}, \vec{w} \in \mathbb{R}^n$:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$$

Another way to calculate the dot product:
Given $\vec{v}, \vec{w}$ with $\angle \vec{v}\vec{w} = \theta$:

$$\vec{v} \cdot \vec{w} = ||\vec{v}|| ||\vec{w}|| \cos \theta$$

1.3.1 Properties of the Dot Product

Given vectors $\vec{u}, \vec{v}, \vec{w}$ and scalar s, the following properties hold:

- Commutative $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
- Associative $(s\vec{v}) \cdot \vec{w} = s(\vec{v} \cdot \vec{w})$
- Distributive $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

1.3.2 Uses of the Dot Product

- From the dot product we can define the length of a vector, $\vec{v}$.

$$||\vec{v}||^2 \equiv \vec{v} \cdot \vec{v}$$

  We can find the length, $||\vec{v}||$, of a three-dimensional vector, $\vec{v} \epsilon \mathbb{R}^3$:

$$||\vec{v}|| = \sqrt{v_1 * v_1 + v_2 * v_2 + v_3 * v_3}$$

- We can define the concept of orthogonality. We say two vectors, $\vec{v}, \vec{w}$ are *orthogonal* (perpendicular) if $\vec{v} \cdot \vec{w} = 0$
- Given vectors, $\vec{v}, \vec{w}$ we can find the projection, $\vec{p}$, of $\vec{w}$ onto $\vec{v}$.

$$\vec{p} = \frac{\vec{v} \cdot \vec{w}}{\vec{v} \cdot \vec{v}} \vec{v}$$

- Given vectors, $\vec{v}, \vec{w}$ and the projection, $\vec{p}$, of $\vec{v}$ onto $\vec{w}$ we can find a vector, $\vec{q}$ orthogonal to $\vec{v}$.

$$\vec{q} = \vec{w} - \vec{p}$$

1.4 Cross Product

Given $\vec{v}, \vec{w} \epsilon \mathbb{R}^3$

$$\vec{v} \times \vec{w} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_1 w_3 - v_3 w_1 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

We can calculate the magnitude of the cross product:
Given $\vec{v}, \vec{w}$ with $\angle \vec{v}\vec{w} = \theta$

$$||\vec{v} \times \vec{w}|| = ||\vec{v}|| ||\vec{w}|| \sin \theta$$

1.4.1 Properties of the Cross Product

Given vectors $\vec{u}, \vec{v}, \vec{w}$ and scalar s, the following properties hold:

- $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
- Distributive $\vec{u} \times (\vec{v} \cdot \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- Associative $(s\vec{v}) \times \vec{w} = s(\vec{v} \times \vec{w})$

1.4.2 Uses of the Cross Product

- Given vectors $\vec{v}, \vec{w}$ we can find a vector $\vec{u}$ which is perpendicular to both $\vec{v}$ and $\vec{w}$.
$$\vec{u} = \vec{v} \times \vec{w}$$
- If the cross product of two vectors is 0, that means the vectors are colinear.

2 Vector Spaces

A *vector space* over $\mathbb{R}^n$ is a set of vectors, for which any vectors $\vec{u}, \vec{v}, \vec{w} \;\epsilon\; \mathbb{R}^n$ and any scalars $r, s \;\epsilon\; \mathbb{R}$ have the following properties:

- Closure under addition:
$$\vec{v} + \vec{w} \;\epsilon\; \mathbb{R}^n$$
- Associative property of addition:
$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$
- Commutative property of addition:
$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$
- Additive identity of addition:
$$\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$$
- Additive inverse:
$$\vec{v} + \vec{-v} = \vec{-v} + \vec{v} = \vec{0}$$
- Closure under scalar multiplication:
$$s\vec{v} \;\epsilon\; \mathbb{R}^n$$
- Associative property of scalar multiplication:
$$r(s\vec{v}) = (rs)\vec{v}$$
- Distributive properties of addition:
$$s(\vec{v} + \vec{w}) = s\vec{v} + s\vec{w} \text{ and } (r + s)\vec{v} = r\vec{v} + s\vec{v}$$
- Identity for scalar multiplication:
$$1\vec{v} = \vec{v}$$

2.1 Some useful vector space definitions

- The vectors $\vec{v}_1, \ldots, \vec{v}_k$ in a vector space V *span a subspace* S if every $\vec{v} \epsilon S$ can be written $\vec{v} = \alpha_1 \vec{v}_1, \ldots, \alpha_k \vec{v}_k$ where $\alpha_i \epsilon \mathbb{R}$ for all $1 \leq i \leq k$.
- The vectors $\vec{v}_1, \ldots, \vec{v}_k$ in a vector space V are *linearly independent* if $\alpha_1 \vec{v}_1, \ldots, \alpha_k \vec{v}_k = 0$ iff $\alpha_i = 0$ for all $1 \leq i \leq k$.
- The vectors $\vec{v}_1, \ldots, \vec{v}_n$ *form a basis* for V if they are linearly independent and they span V.
- The *dimension* of a vector space V is the number of vectors in any basis for V.
- A collection of vectors $\vec{v}_1, \ldots, \vec{v}_k$ is *mutually orthogonal* if each pair $\vec{v}_i, \vec{v}_j$ with $i \neq j$ is orthogonal.
- A basis consisting of mutually orthogonal vectors is called an *orthogonal* basis. If the vectors are also unit vectors then the basis is called an *orthonormal* basis.

3 Matrix Operations

3.1 Addition

Given $A, B \epsilon \mathbb{R}^{m \times n}$:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & \ldots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & \ldots & a_{2n} + b_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \ldots & a_{mn} + b_{mn} \end{bmatrix}$$

3.2 Scalar Multiplication

Given $A \epsilon \mathbb{R}^{m \times n}$ and $s \epsilon \mathbb{R}$:

$$sA = \begin{bmatrix} sa_{11} & \ldots & sa_{1n} \\ sa_{21} & \ldots & sa_{2n} \\ \vdots & \ddots & \vdots \\ sa_{m1} & \ldots & sa_{mn} \end{bmatrix}$$

3.3 Vector Multiplication

Given $A \epsilon \mathbb{R}^{m \times n}$ and $\vec{v} \epsilon \mathbb{R}^n$:

$$A\vec{v} = \begin{bmatrix} a_{11}v_1 + \ldots + a_{1n}v_n \\ a_{21}v_1 + \ldots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + \ldots + a_{mn}v_n \end{bmatrix}$$