Computation of Discrete Function Chrestenson Spectrum Using Cayley Color Graphs*

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Abstract

A method based on eigenvalue computations is formulated for computing the Chrestenson spectrum of a discrete p-valued function. This technique is developed by first considering an extension to the conventional approach to computing the Walsh spectrum for a binary-valued function which is then generalized to the p-valued case (where $p > 2$). Algebraic groups are formulated that correspond to Cayley color graphs based on the function of interest. These graphs have spectra equivalent to the Walsh or Chrestenson spectrum of the function under consideration. Because the transformation matrix is not used in any of these computations, the method provides an alternative approach for spectral computations. This work also illustrates the correspondence between algebraic group theory and discrete logic function spectral methods.

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1 Introduction

In [2] a method for computing the Walsh spectrum in R-encoding of a completely specified binary-valued function was presented based on the formation of an algebraic group dependent on the binary-valued function yielding a corresponding Cayley graph [6, 7, 28]. It is proven in [2] that the spectrum of the Cayley graph is in fact equivalent to the Walsh spectrum of the Boolean function using algebraic group character theory [12]. Some extensions and applications of this approach are provided in [3, 4, 27].

A generalization of the set of orthogonal Walsh functions over the binary field leads to the set of Chrestenson functions [8]. These functions may be used as basis functions for a discrete transform of a p-valued function. Several researchers have investigated the computation and application of the Chrestenson spectrum for p-valued discrete functions and in particular, the ternary case [21, 22, 25]. Applications that have been considered in the past are synthesis [11, 16], decomposition [23, 26] and testing [10, 20].

In the following it is shown that the spectrum of a Cayley color graph formulated from an algebraic group similar to that used in [2] yields the Walsh spectrum in S-encoding for a fully specified binary-valued function. These results are then generalized and it is shown that the Chrestenson spectrum of a fully specified p-valued function may likewise be computed as the spectrum of a Cayley color graph. A specific example is given for the case of a ternary-valued function.

The remainder of this paper is organized as follows. Section 2 will provide a brief overview of the method described in [2] and extensions are given and proven for binary-valued functions. In Section 3 the computation of the Chrestenson spectrum of a ternary-valued function is reviewed and it is shown how the techniques for the binary-valued case can be extended to the multiple-valued case. In Section 4, an example computation is given showing how the method described here can be used. Conclusions are provided in Section 5.

2 Extensions for Binary-valued Functions

In this Section notation and definitions are provided that will be used throughout the paper followed by extensions to the spectral computation technique described in [2]. The method described and proven in [2] produces the Walsh spectrum of a binary-valued function and extensions to this technique are also given. An alternative proof is also provided based on linear algebra arguments.

2.1 Eigenvalue Technique - Binary Case

Definition 1 *A Cayley graph can be considered a graphical representation of a finite algebraic group. As an example, consider a group* $(M, *)$ *and a set* N *which is a subset of* M*. The Cayley graph of* M *with respect to* N *can be denoted as* $Cay(M, N)$*. The vertex set of* $Cay(M, N)$ *consists of the elements of* M *and the edge set consists of pairs of elements from* M*,* (x, y) *such that* $y = x \diamond n$ *for some* $n \in N$*. Note that the group product operation* $*$ *and the generator operation* $\diamond$ *are not necessarily the same.*

In [2] a group is formulated and described by $(M, \oplus)$ where $M = \{m_i\}$ is the set of all possible 2^n minterms for a Boolean function f and $\oplus$ is the exclusive-OR operation applied bit by bit over a pair (m_i, m_j). This group can be used with the function of interest f to form a Cayley graph. The $2^n \times 2^n$ adjacency matrix $\mathbf{A} = [\mathbf{a_{ij}}]$ of the Cayley graph is formed using the relationship given in Equation 1.

$$a_{ij} = f(m_i \oplus m_j) \tag{1}$$

Since Equation 1 evaluates to the Boolean constants 0 or 1 the resulting undirected graph contains non-weighted edges and yields a Cayley graph. As proven in [2], the spectrum of the graph, defined as the set of eigenvalues of $\mathbf{A}$ [5, 9], is also the R-encoded Walsh spectrum. R-encoding refers to mapping function values of Boolean 0 to the integer 0 and Boolean 1 to the integer +1.

Remark 1 *The adjacency matrix* $\mathbf{A}$ *has the special property that the sum of each row and each column has the same value* q*. The* $\mathbf{A}$ *matrix is a block-circulant matrix since it describes the group* M*.*

Lemma 1 *The Cayley graph formed using the group* M *and equation 1 consists of vertices that either all have self-loops or where none have self-loops. The diagonal elements of* $\mathbf{A}$ *are equivalent to* $f(0)$*.*

Proof: The diagonal elements $a_{ii} \in \mathbf{A}$ are equivalent to $f(0)$ since, from Equation 1, $a_{ii} = f(m_i \oplus m_i) = f(0)$. When $f(0) = 1$ self-loops are present and the diagonal of $\mathbf{A}$ is all ones, otherwise no self-loops are present. $\square$

Definition 2 *A Hadamard matrix is composed of elements that are all +1 or -1 and has row and column vectors that are orthogonal [14]. A Hadamard matrix whose row vectors are discretized Walsh functions and has order $2^n \times 2^n$ is denoted by $\mathbf{H}$. The naturally ordered Hadamard matrix $\mathbf{H}$ may be defined as shown in Equations 2 and 3 where $\otimes$ represents the Kronecker product.*

$$\mathbf{H_0} = [\mathbf{1}] \tag{2}$$

$$\mathbf{H_n} = \begin{bmatrix} \mathbf{H_{n-1}} & \mathbf{H_{n-1}} \\ \mathbf{H_{n-1}} & -\mathbf{H_{n-1}} \end{bmatrix} = \bigotimes_{\mathbf{i=1}}^{\mathbf{n}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{3}$$

The $\mathbf{H}$ matrix has several useful properties such as symmetry about the diagonal, $\mathbf{H} = \mathbf{H^T}$. Because $\mathbf{H}$ is orthogonal it follows that $\mathbf{H^{-1}} = \frac{\mathbf{1}}{\mathbf{2^n}}\mathbf{H}$.

Definition 3 *The Walsh spectrum of a binary-valued function is defined as the vector $\mathbf{s}$ composed of elements s_i where each s_i is termed a spectral coefficient. The Walsh spectral vector can be calculated as a linear transformation of the function vector $\mathbf{f}$ using the matrix $\mathbf{H}$ as $\mathbf{s} = \mathbf{Hf}$.*

Definition 4 *A matrix $\mathbf{D} = [\mathbf{d_{ij}}]$ is defined as a diagonal matrix such that each $d_{ij} = 0 \forall i \neq j$ and $d_{ii} = s_i$. Another defintion is $\mathbf{D} = \mathbf{s^T I}$.*

Definition 5 *If a similarity matrix $\mathbf{S}$ exists such that $\mathbf{S^{-1}PS} = \mathbf{Q}$ where $\mathbf{P}$ is an arbitrary matrix and $\mathbf{Q}$ is a diagonal matrix, then $\mathbf{P}$ is said to be a diagonalizable matrix.*

Definition 6 *If two arbitrary matrices $\mathbf{P}$ and $\mathbf{T}$ are both diagonalizable using an identical similarity matrix $\mathbf{S}$, then $\mathbf{P}$ and $\mathbf{T}$ are said to be simultaneously diagonalizable.*

Lemma 2 *The matrices $\mathbf{A}$ and $\mathbf{D}$ are similar matrices [18].*

Lemma 3 *Since* $\mathbf{A}$ *and* $\mathbf{D}$ *are similar they have the same set of eigenvalues with corresponding multiplicities. Stated in alternative terminology,* $\mathbf{A}$ *and* $\mathbf{D}$ *have identical spectra.*

Theorem 1 *The eigenvalues of* $\mathbf{A}$ *are the Walsh spectral coefficients of the binary valued function* f*.*

Proof: Since $\mathbf{D}$ is a diagonal matrix it is obvious that the set of eigenvalues for $\mathbf{D}$ are $\{d_{ii}\}$. By definition $d_{ii} = s_i$, thus the eigenvalues of $\mathbf{D}$ are the spectral coefficients of f.

From Lemma 2 it is shown that $\mathbf{A}$ and $\mathbf{D}$ are simultaneously diagonalizable and from Lemma 3 it is shown that $\mathbf{A}$ and $\mathbf{D}$ necessarily have identical eigenvalues with corresponding multiplicities and it is concluded that the eigenvalues of $\mathbf{A}$ are the set $\{s_i\}$. □

It is noted that a similar proof of this result based on group character theory with respect to general Cayley graph spectra is given in [1] and later with respect to binary-valued functions using the specific group M in [2].

The following theorem appeared in [18]. It is included here for completeness.

Theorem 2 *The similarity matrix* $\mathbf{S}$ *satisfying* $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$ *is in fact the Hadamard matrix* $\mathbf{H}$*.*

Theorem 2 leads to the useful relationship given in Equation 4.

$$\frac{1}{2^n}\mathbf{H}\mathbf{A}\mathbf{H} = \mathbf{D} \tag{4}$$

Additionally, due to Remark 1 and Theorem 2 it is seen that the eigenvalues of the adjacency matrix $\mathbf{A}$ must be the Walsh spectral coefficients of the binary-valued function f since the eigenvectors of circulant matrices correspond to row vectors in a Hadamard matrix.

In [27] an extension of this method is presented where it is shown that the Walsh spectrum of the complement of f, $\overline{f}$, results when the group $(M, \equiv)$ is used where $\equiv$ is the bitwise equivalence operator (the exclusive-NOR). It is easy to see this is the case since the equivalence operator produces the complement of f.

In contrast to R-encoding as is used in [2], it is also possible to generate the Walsh coefficients of a binary-valued function as a computation of the eigenvalues of an S-encoded adjacency matrix as described in [27]. An S-encoded spectrum results when the elements of $\mathbf{A}$ are mapped to the integer values +1 and -1. The mapping of the $\mathbf{A}$ elements to the two roots of unity $e^{j0} = +1$ and $e^{j\pi} = -1$ result in a Cayley color graph. The Cayley color graph is a completely connected graph with edge weights (i.e. colors) corresponding to +1 for a Boolean-0 constant and -1 for a Boolean-1 constant. As described in [17, 24] conversion from R to S encoded spectra can be accomplished as given in Equation 5.

$$s_i = \begin{cases} 2^n - 2r_0, & i = 0 \\ -2r_i, & i > 0 \end{cases} \tag{5}$$

R-encoding represents a mapping of function range values such that the Boolean constant 0 is mapped to the integer 0 and the Boolean constant 1 is mapped to the integer +1. S-encoding represents the mapping of the binary Boolean constants to the complex plane as two roots of unity. This mapping is illustrated in Figure 1 for binary valued functions.

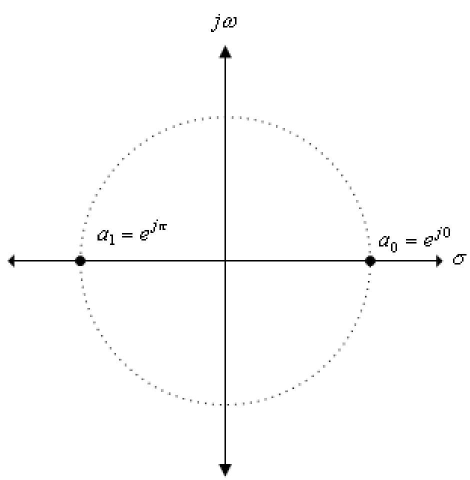

Figure 1: Complex Plane with Encoded Binary Function Values

Lemma 4 *Let* $\mathbf{J}$ *denote the* $k \times k$ *matrix of all 1's. Then the eigenvalues of* $\mathbf{J}$ *are* k *(with multiplicity one) and 0 (with multiplicity k-1).*

Proof: Since all rows are equal and nonzero, rank(J)=1. Since a $k \times k$ matrix of rank$(k-h)$ has at least h eigenvalues equal to 0, it is concluded that $\mathbf{J}$ has at least $k-1$ eigenvalues equal to 0. Since trace$(\mathbf{J}) = \mathbf{k}$ and the trace is the sum of the eigenvalues, it follows that the remaining eigenvalue of $\mathbf{J}$ is equal to k. □

Lemma 5 *The all 1's $k \times k$ matrix $\mathbf{J}$ and the adjacency matrix $\mathbf{A}$ defined by Equation 1 are simultaneously diagonalizable.*

Proof: From [15] it is proven that $\mathbf{J}$ and $\mathbf{A}$ are simultaneously diagonalizable if, and only if, they commute under matrix multiplication. Due to the definition of $\mathbf{A}$ in Equation 1, it is seen that $\mathbf{A} = \mathbf{A^T}$. It follows that $\mathbf{AJ} = \mathbf{A^T J} = \mathbf{A^T J^T} = (\mathbf{JA})^\mathbf{T}$. From Remark 1 it is noted that the sum of all rows and columns of $\mathbf{A}$ is q, Thus, $\mathbf{AJ} = q\mathbf{J} = (q\mathbf{J})^\mathbf{T} = (\mathbf{AJ})^\mathbf{T}$. Equating the previous two expressions yields $(\mathbf{JA})^\mathbf{T} = (\mathbf{AJ})^\mathbf{T}$. Therefore, $\mathbf{AJ} = \mathbf{JA}$ thus $\mathbf{A}$ and $\mathbf{J}$ are simultaneously diagonalizable. □

Definition 7 *The matrix $\mathbf{B}$ is the S-encoded adjacency matrix $\mathbf{A}$ where all 0-valued components of $\mathbf{A}$ are replaced by +1 and all 1-valued components are replaced by -1.*

Theorem 3 *The eigenvalues of $\mathbf{B}$ are the S-encoded Walsh spectral coefficients of the binary function used to formulate the $\mathbf{A}$ matrix corresponding to $\mathbf{B}$.*

Proof: The $2^n \times 2^n$ $\mathbf{B}$ matrix is related to $\mathbf{A}$ as shown in Equation 6 where the $2^n \times 2^n$ $\mathbf{J}$ matrix is one with all elements equal to +1.

$$\mathbf{B} = \mathbf{J} - \mathbf{2A} \tag{6}$$

From Lemma 5, $\mathbf{J}$ and $\mathbf{A}$ are simultaneously diagonalizable and a non-singular similarity matrix $\mathbf{S_m}$ exists that diagonalizes each. The diagonal matrices $\mathbf{D_J}$ and $\mathbf{D_A}$ corresponding to the similarity transform matrix $\mathbf{S_m}$ are given in Equations 7 and 8.

$$\mathbf{D_J} = \mathbf{S_m^{-1} J S_m} \tag{7}$$

$$\mathbf{D_A} = \mathbf{S_m^{-1} A S_m} \tag{8}$$

From Equations 7 and 8, it is observed that another diagonal matrix, $\mathbf{D_B}$ results as follows.

$\mathbf{D_B} = \mathbf{D_J} - \mathbf{2D_A} = \mathbf{S_m^{-1}}(\mathbf{J} - \mathbf{2A})\mathbf{S_m} = \mathbf{S_m^{-1}JS_m} - \mathbf{2S_m^{-1}AS_m}$

The eigenvalues of $\mathbf{B}$ and $\mathbf{D_B}$ are the same due to the existence of the similarity transform $\mathbf{S_m}$. Thus, the eigenvalues of $\mathbf{B} = \mathbf{J} - \mathbf{2A}$ are of the form $\lambda_i - 2\mu_j$ where $\{\lambda_i\}$ and $\{\mu_i\}$ are the eigenvalues of $\mathbf{J}$ and $\mathbf{A}$ respectively. From Lemma 4 it is seen that the eigenvalues of $\mathbf{J}$ are $\lambda_0 = 2^n$ with all remaining $\lambda_i = 0$ for all $i \neq 0$.

Since $e = (1, 1, \ldots, 1)^T$ is an eigenvector of $\mathbf{J}$ corresponding to the eigenvalue $\lambda_0 = 2^n$ and μ_0 is equal to the number of 1's in a row of $\mathbf{A}$ is an eigenvalue of $\mathbf{A}$, one eigenvalue of $\mathbf{B} = \mathbf{J} - \mathbf{2A}$ is $2^n - 2\mu_0$ which is of the form of s_0 as given in Equation 5. All other eigenvalues must be of the form $0 - 2\mu_j$ where $j \neq 0$ yielding the remaining Walsh spectral coefficients, s_i, as given in Equation 5. $\square$

3 Spectra of p-valued Functions Using Cayley Color Graphs

The computation of the Chrestenson spectrum of a fully specified ternary-valued discrete function using the mathematically defined methods such as those described in [19] is briefly reviewed. Following the review extensions of the techniques described in Section 2 are applied to the computation of p-valued functions. Although ternary-valued functions are used as examples, these techniques are generalizable to any p-valued function.

3.1 Chrestenson Spectrum of Ternary Function

By representing all discrete function values as roots of unity in the complex plane as shown in Figure 2 for ternary valued functions, a direct mapping to spectral representations results. In this mapping, the ternary logic values of $\{0, 1, 2\}$ are mapped to values in the complex plane $\{a_0, a_1, a_2\}$ respectively where $a_0 = e^{j0} = 1$, $a_1 = e^{j\frac{2\cdot\pi}{3}}$ and $a_2 = e^{j\frac{4\cdot\pi}{3}}$.

To compute the Chrestenson spectrum of a ternary-valued function, all logic-0 values are mapped to a_0, logic-1 values to $a_1^* = a_2$ and logic-2 values to $a_2^* = a_1$. The superscript $*$ denotes the complex conjugate of the

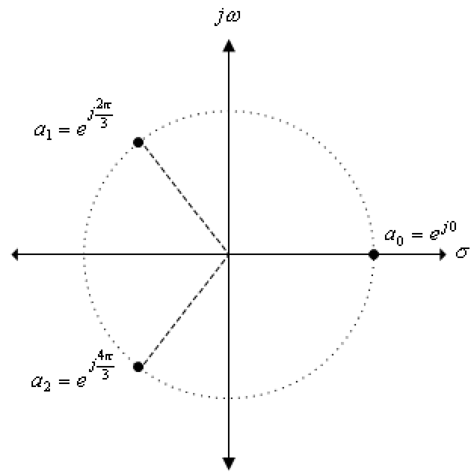

Figure 2: Complex Plane with Encoded Ternary Function Values

values that correspond to the original function values. This is necessary for the inverse transform to yield the proper values. The resulting vector of complex values is then linearly transformed as a matrix-vector product. The Chrestenson transformation matrix may be formulated in natural order using the Kronecker product definition [13] as shown in Equations 9 and 10.

$$\mathbf{C^0} = [\mathbf{1}] \tag{9}$$

$$\mathbf{C^n} = \bigotimes_{\mathbf{i=1}}^{\mathbf{n}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a_1 & a_2 \\ 1 & a_2 & a_1 \end{bmatrix} \tag{10}$$

As an example, the function f defined by the truth table shown in Table 1 is transformed using the transformation matrix in Equation 11. In Table 1 the fourth column, f_{ENC}, contains the function values after encoding into the complex conjugates of the roots of unity shown in Figure 2.

Table 1: Truth Table of Example Ternary Function

x_1	x_2	f	f_{ENC}
0	0	0	a_0
0	1	2	a_1
0	2	0	a_0
1	0	1	a_2
1	1	0	a_0
1	2	2	a_1
2	0	2	a_1
2	1	1	a_2
2	2	0	a_0

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a_1 & a_2 & 1 & a_1 & a_2 & 1 & a_1 & a_2 \\ 1 & a_2 & a_1 & 1 & a_2 & a_1 & 1 & a_2 & a_1 \\ 1 & 1 & 1 & a_1 & a_1 & a_1 & a_2 & a_2 & a_2 \\ 1 & a_1 & a_2 & a_1 & a_2 & 1 & a_2 & 1 & a_1 \\ 1 & a_2 & a_1 & a_1 & 1 & a_2 & a_2 & a_1 & 1 \\ 1 & 1 & 1 & a_2 & a_2 & a_2 & a_1 & a_1 & a_1 \\ 1 & a_1 & a_2 & a_2 & 1 & a_1 & a_1 & a_2 & 1 \\ 1 & a_2 & a_1 & a_2 & a_1 & 1 & a_1 & 1 & a_2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_0 \\ a_2 \\ a_0 \\ a_1 \\ a_1 \\ a_2 \\ a_0 \end{bmatrix} = \begin{bmatrix} 4a_0 + 3a_1 + 2a_2 \\ 3a_0 + 2a_1 + 4a_2 \\ 2a_0 + 4a_1 + 3a_2 \\ 4a_0 + 3a_1 + 2a_2 \\ 3a_0 + 2a_1 + 4a_2 \\ 8a_0 + a_1 \\ 4a_0 + 3a_1 + 2a_2 \\ 3a_0 + 2a_1 + 4a_2 \\ 2a_0 + 4a_1 + 3a_2 \end{bmatrix} \quad (11)$$

3.2 Cayley Color Graph Spectrum for p-valued Functions

To extend the Cayley color graph spectrum computation method for Chrestenson spectrum calculations for fully specified p-valued functions, it is necessary to formulate an appropriate algebraic group. The $\oplus$ operator used for the case of binary valued functions can be viewed as an arithmetic operation in $GF(2)$. With this point of view $\oplus$ can be considered as addition or subtraction modulo-2 over the set of elements in M. While the use of the addition operation modulo-p as a generator operation does indeed result in a Cayley color graph [28], the spectrum of the graph does not result in the corresponding Chrestenson spectrum for the function of interest. This is easy to see since the diagonal of the resulting adjacency matrix $\mathbf{A}$ does not contain same valued elements as is the case for binary-valued functions; however, if the generator operator $\ominus_p$ is used where $\ominus_p$ represents the difference modulo-p, this characteristic is preserved. Hence, as is analogous to the technique for binary-valued functions, the elementary Abelian p-adic addition group $(M, \oplus_p)$ may be used where $\oplus_p$ represents addition modulo-p. The Cayley graph is formed using the $\ominus_p$ generator operation and the corresponding adjacency matrix for the discrete p-valued function is likewise formed with each component defined by the relationship in Equation 12 where each a_{ij} is colored by the mapping defined above.

$$a_{ij} = f(m_i \ominus_p m_j) \tag{12}$$

3.3 Theoretical Basis for the Spectrum Computation of p-valued Functions

Finite algebraic groups may be characterized using character theory [12]. A character table of a group can be formulated such that the columns are labeled by conjugacy classes and the rows are labeled by the irreducible characters χ_i. The conjugacy classes represent the various p roots of unity where the 0^{th} power of the root is the real value 1. These values are members of the set of complex values. As an example, consider the character table for the cyclic group of order 2 as shown in Table 2.

Table 2: Character Table for $p = 2$

classes	e^{j0}	$e^{j\pi}$
χ_1	1	1
χ_2	1	-1

It is noted that the character table shown in Table 2 is a representation of a Hadamard matrix as described in [14]. From the results of Babai's work in [1] the spectral coefficients of the corresponding Cayley graph representing the group as described in Equation 12 is the vector scalar product of the rows of the character table in 2 with the function represented by the group relationship. It is noted that this is the definition of the S-encoded Walsh spectrum of a binary-valued function as described in [17, 24]. These results are generalized in [1] and apply to values of $p > 2$.

In keeping with the ternary-valued example described in this paper, the following character table as shown in Table 3 is given.

Table 3: Character Table for $p = 3$

classes	$e^{j0} = a_0$	$e^{j\frac{2\pi}{3}} = a_1$	$e^{j\frac{4\pi}{3}} = a_2$
χ_1	a_0	a_0	a_0
χ_2	a_0	a_1	a_2
χ_3	a_0	a_2	a_1

As is proven in [1], a spectral coefficient of the representative Cayley graph of the discrete function under consideration is the dot product of the colors represented in the character table with the discrete function truth vector. It is apparent that the character table as shown in Table 3 is the same as the transformation matrix given in Equation 9. Thus, the relationship between the linear Chrestenson transform, the corresponding Cayley color graph and the group character table is illustrated.

In [16] the concept of a ternary-quadrature Chrestenson spectrum is introduced where the values (-1,j,+1) are used in the formulation of the transformation matrix rows instead of (a_0, a_1, a_2). The eigenvalue method described here will not work for this case since (-1,j,+1) are not roots of unity. However, the concept of an R-encoded Chrestenson spectrum as also described in [16] can be used with this technique. The R-encoded Chrestenson spectrum is one where the function values are not encoded, only the transformation matrix.

4 Example Calculation

As an example of the extension of the Cayley color graph method for computing the Chrestenson spectrum of a fully specified p-valued function, consider the ternary-valued function (i.e. $p = 3$) defined by the truth table given in Table 1. Using the group and corresponding **A** matrix defined in Equation 12 the adjacency matrix given in Equation 13 results. Note that the entries of the adjacency matrix are in fact the complex conjugates of the function values when mapped to the roots of unity as shown in Figure 2. If complex conjugates are not used in the formation of **A**, then the resulting eigenvalues of **A** are the complex conjugates of the spectrum [15].

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & a_1 & a_1 & 1 & a_2 & a_2 & a_1 & 1 \\ a_1 & 1 & 1 & a_2 & a_1 & 1 & 1 & a_2 & a_1 \\ 1 & a_1 & 1 & 1 & a_2 & a_1 & a_1 & 1 & a_2 \\ a_2 & a_1 & 1 & 1 & 1 & a_1 & a_1 & 1 & a_2 \\ 1 & a_2 & a_1 & a_1 & 1 & 1 & a_2 & a_1 & 1 \\ a_1 & 1 & a_2 & 1 & a_1 & 1 & 1 & a_2 & a_1 \\ a_1 & 1 & a_2 & a_2 & a_1 & 1 & 1 & 1 & a_1 \\ a_2 & a_1 & 1 & 1 & a_2 & a_1 & a_1 & 1 & 1 \\ 1 & a_2 & a_1 & a_1 & 1 & a_2 & 1 & a_1 & 1 \end{bmatrix} \tag{13}$$

The adjacency matrix given in Equation 13 has several structural properties typical of Cayley color graphs. **A** is a matrix with all values along the diagonal equivalent to the value of the function at the all-zero minterm $m_0 = 00\ldots0$. Also, the first column of **A** corresponds to the complex conjugate values of the function when mapped to the complex plane. When **A** is considered in terms of all $p \times p$ submatrix blocks (in this case $p = 3$), it is seen that **A** is a block circulant matrix with each submatrix being circulant. It is also noted that the matrix $\mathbf{A^T}$ can likewise be used to compute the Chrestenson spectrum since the eigenvalues of **A** are equivalent to those of $\mathbf{A^T}$ [15]. Obtaining **A** or $\mathbf{A^T}$ corresponds to taking the difference modulo-p of minterm values labeling the rows of **A** with those of the column or vice versa.

Computing the eigenvalues of **A** in Equation 13 results in the set of values $\lambda_i = (4a_0 + 3a_1 + 2a_2, 3a_0 + 2a_1 + 4a_2, 2a_0 + 4a_1 + 3a_2, 4a_0 + 3a_1 + 2a_2, 3a_0 + 2a_1 + 4a_2, 8a_0 + a_1, 4a_0 + 3a_1 + 2a_2, 3a_0 + 2a_1 + 4a_2, 2a_0 + 4a_1 + 3a_2)$ which are also the Chrestenson spectral coefficients as can be verified by comparison to Equation 11 where the spectrum is computed according to the definition.

5 Conclusions

The computation of the Chrestenson spectrum of a p-valued discrete function is shown to be accomplished by computing the spectrum of an adjacency matrix representing a Cayley color graph. The matrix is formed as a collection of column (or row) vectors each of which is a permutation of the ternary function truth vector when encoded in the complex plane. These results are of theoretical interest since they illustrate the relationship between the spectra of a multi-valued discrete functions and the eigenvalue computation problem.

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